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Interaction between different scales of turbulence over short times

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The distortion by large-scale random motions of small-scale turbulence is investigated by examining a model problem. The changes in energy spectra, velocity and vorticity moments, and anisotropy of small-scale turbulence are calculated over timescales short compared with the timescale of small-scale turbulence by applying rapid distortion theory with a random distortion matrix for different initial conditions: irrotational or rotational, and isotropic or anisotropic large-scale turbulence with or without mean strain, and isotropic or anisotropic small-scale turbulence.

We have obtained the following results: (1) Irrotational random strains broaden the small-scale energy spectrum and transfer energy to higher wavenumbers. (2) The rotational part of the large-scale strain is important for reducing anisotropy of turbulence rather than transferring energy to higher wavenumbers. (3) Anisotropy in small-scale turbulence is reduced by large-scale isotropic turbulence. The reduction of anisotropy of the velocity field depends on the initial value of the velocity anisotropy tensor of the small-scale velocity field u_i defined by $\overline{u_i u_j}/\overline{u_l u_l} - \frac{1}{3}\delta_{ij}$, and also on the anisotropy of the distribution of the energy spectrum in wavenumber space. The reduction in anisotropy of the vorticity field ω_i depends only on the vorticity anisotropy tensor. (4) The pressure–strain correlation is calculated for the change in Reynolds stress of the anisotropic small-scale turbulence. The correlation is proportional to time and depends on the difference between the velocity and wavenumber anisotropy tensors. These results (which are exact for small time) differ significantly from current turbulence models. (5) The effect of large-scale anisotropic turbulence on isotropic small-scale turbulence is calculated in general. Results are given for the case of axisymmetric large scales and are compared with the observed behaviour of small-scale turbulence near interfaces. (6) When a mean irrotational straining motion is applied to turbulence with distinct large-scale and small-scale components in their velocity field, the large-scale irrotational motions combine with the mean straining to increase further the anisotropy of the vorticity of the small scales, but the large-scale rotational motions reduce the small-scale anisotropy. For isotropic straining motion, the latter is weaker than the former. After the mean distortion ceases, both kinds of large-scale straining tend to reduce the anisotropy. This also has implications for modelling the rate of reduction of anisotropy.

1. Introduction

An essential feature of turbulence is the interaction between motions at different scales. This interaction gives rise to the transfer of energy, which is usually (but not

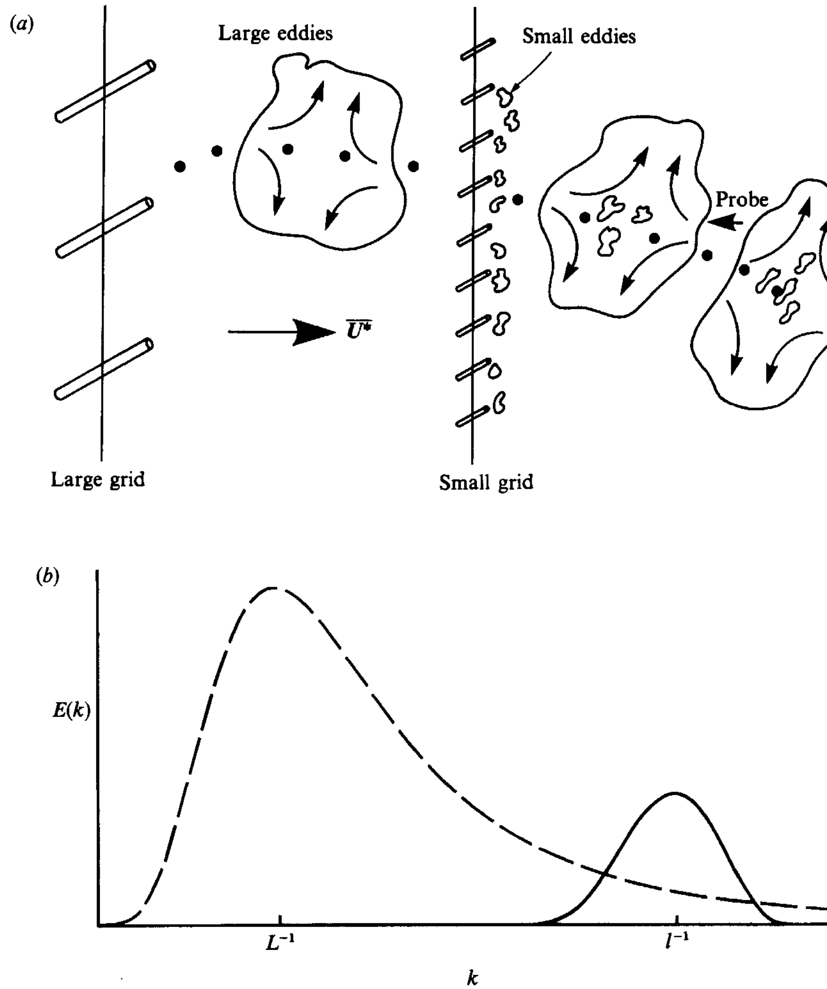

FIGURE 1 (*a*, *b*). For caption see facing page.

always) from large to small scales in three-dimensional turbulence, or from small to large scales in two-dimensional turbulence. It also affects the tendency towards isotropy of turbulence (though how it does so is not understood), and determines the manner of the response of the turbulence to gradients in the mean flow or changes in the boundary conditions, both when the response is rapid as in wind-tunnel contractions or slow as in shear flows. These processes are incorporated more or less implicitly in approximate models of turbulence ranging from theories modelling the full two-point spectra such as Edwards (1964), Kraichnan (1959) and EDQNM (e.g. Lesieur 1987), to Reynolds stress closures such as those of Lumley (1978), Launder, Reece & Rodi (1975) and Cambon, Jeandel & Mathieu (1981). In Edwards' model there is an explicit hypothesis of the relaxation time $\tau(k)$ for each part of wave-number space, corresponding to these interactions.

There seems to be two ways of examining these interactions explicitly: either to devise experiments, such as those of Kellogg & Corrsin (1980), Itsweire & Van Atta (1984) and Tan-atichat, Nagib & Loehrke (1982); or to perform computations using the full equations of motion on flows where turbulence is generated at two distinct

(c)

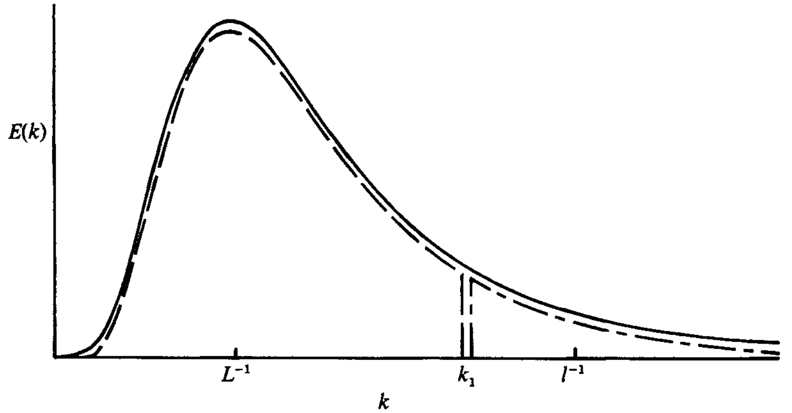

FIGURE 1. Aspects of the study of the interaction of large- and small-scale turbulence. (*a*) Turbulence generated by a large grid interacting with small-scale turbulence generated by a small grid. Note how the small eddies are distorted by the large eddies – that is what we calculate. (*b*) The energy spectra of large- and small-scale turbulent velocity fields with a distinct scale separation between them – these spectra might be based on the whole record at a point or based on a conditionally sampled record such as within a coherent structure. (*c*) The transfer of energy between scales less and greater than k_1^{-1} in a continuous spectrum (——) is often analysed by assuming that the spectrum for $k < k_1$, (– – –) can be regarded as separated from the spectrum $k > k_1$ (– – – –). This diagram indicates the idealization.

scales; for example, large-scale turbulence produced by a grid in a wind tunnel might pass through another grid with a different scale (figure 1*a*). The results of these or similar experiments cannot in general be compared quantitatively with the general theories of turbulence, or even explained in qualitative terms. For example, the second grid may produce broad- or narrow-band velocity fluctuations or merely a non-uniform perturbation on the mean flow; depending on which of these processes is dominant, the interaction with the initial turbulence differs considerably, and different kinds of theory are appropriate. It is certainly not possible to use such experiments to understand the general problem of interaction of small and large scales in turbulence without the development of some theory closely related to these actual experiments or computations.

Another way to examine these interactions explicitly is to analyse or compute an idealized form of these experiments by considering how a small-scale homogeneous turbulent velocity field $\boldsymbol{u}(\boldsymbol{x},t)$ interacts with a primary large-scale random velocity field $U(\boldsymbol{x},t)$. In some cases this interaction may take place in the presence of a mean velocity field $\bar{U}(\boldsymbol{x},t)$. This idealization is used in the EDQNM model (Bertoglio 1986) and in Townsend's (1976, p. 99) estimate of transfer to small scales (figure 1*b*, *c*). For cases where the small-scale turbulence is more energetic than the large-scale turbulence, Pouquet, Frisch & Chollet (1983) have shown that the small scale acts like an eddy viscosity on the large scale, cutting off its high-wavenumber spectrum, while the large scale does not affect the small scales much.

In this paper we concentrate on the situation where the small-scale turbulence is less energetic than the large scale. With suitable generalizations the techniques of rapid distortion theory (r.d.t.) can be applied to predict this interaction over a short time t_{D}, provided that (*a*) the scale of the large-scale turbulence L is much greater than that of the small-scale turbulence l, i.e.

$$L \gg l, \tag{1.1}$$

and (*b*) the rate of straining of the vorticity of the small-scale turbulence by the large-scale turbulence U'/L is much greater than the self-induced straining rate of the small-scale turbulence by itself, u'/l. Here L and l are the lengthscales and U', u' the r.m.s. velocity scales of the large- and small-scale turbulence, i.e.

$$\frac{U'}{L} \gg \frac{u'}{l}. \tag{1.2}$$

Even if these two timescales are of the same order, it is found by computations (e.g. Bertoglio 1986) and experiments that the theory can still approximately describe the distortion because the small-scale distortion is randomly distributed over a period of the distortion and therefore has less net effect than the large scale. In most laboratory turbulent flows at moderate Reynolds number, it is found that the timescales (l/u') of the small scales are comparable with the timescales of the large scales L/U'. At the Reynolds number characteristic of most numerical simulations, even condition (1.2) is satisfied (Hunt, Wray & Buell 1988).

The methods of r.d.t. have been developed to calculate the change of the small scale caused by a uniform large-scale straining motion $\boldsymbol{U} = \boldsymbol{\alpha}\cdot\boldsymbol{x}$, over a time interval $t_D < l/u'$. For each such straining the small-scale turbulence can be averaged spatially over many of its lengthscales l. Then to calculate the interaction of the small-scale turbulence and large-scale turbulence, we simply average over an ensemble of values of $\boldsymbol{\alpha}$.

Examples of straining in different situations are illustrated in figure 2(*a*).

Our analysis is directed towards helping answer some of the following specific questions about the interaction of large and small scales that have been raised, not only in the context of the wind-tunnel experiments referred to earlier, but also in the context of wave–turbulence interaction (Finnigan & Einaudi 1981; Carruthers & Hunt 1988) and distortion of turbulence by rigid surfaces (Hunt 1984).

(1) How does the transfer of energy between small and large scales depend on the relative energies, scales, isotropy and rotationality of the motions? How accurate is Townsend's (1976, p. 99) estimate for the transfer of energy in the inertial subrange, based on an assumption of irrotational straining by the large scales and that the large lengthscales are only slightly larger than the small lengthscales? (See also Kerr 1985.)

(2) How do interactions between large- and small-scale turbulence affect the tendency to isotropy or anisotropy of turbulence? For example, if the small-scale turbulence is anisotropic, how rapidly does large-scale turbulence make it isotropic (cf. Kellogg & Corrsin 1980), or vice versa (Hunt 1984)?

(3) In turbulent flows undergoing distortion by a mean straining motion, how does this affect the interaction between large and small scales? Or as the vortex lines are stretched by the mean motion, how are they distorted by the turbulence itself (figure 2*b*)? After elongation and compression by the mean straining, a small amount of additional random distortion may have a large effect on the vorticity component in the direction in which the mean flow compresses vortex elements (Hunt 1973, p. 661). (In other words, the distortion matrix is randomly rotated and stretched.) This nonlinear effect is the reason why r.d.t. over-predicts the reduction in the variance of one component of turbulence caused by irrotational distortion and is a better approximation for the components that are amplified. We can explore this question by considering the effects of a mean distortion of a turbulent flow consisting of large- and small-scale components, and help to elucidate the limits of r.d.t.

The analysis presented here is a rational calculation, in that the approximations

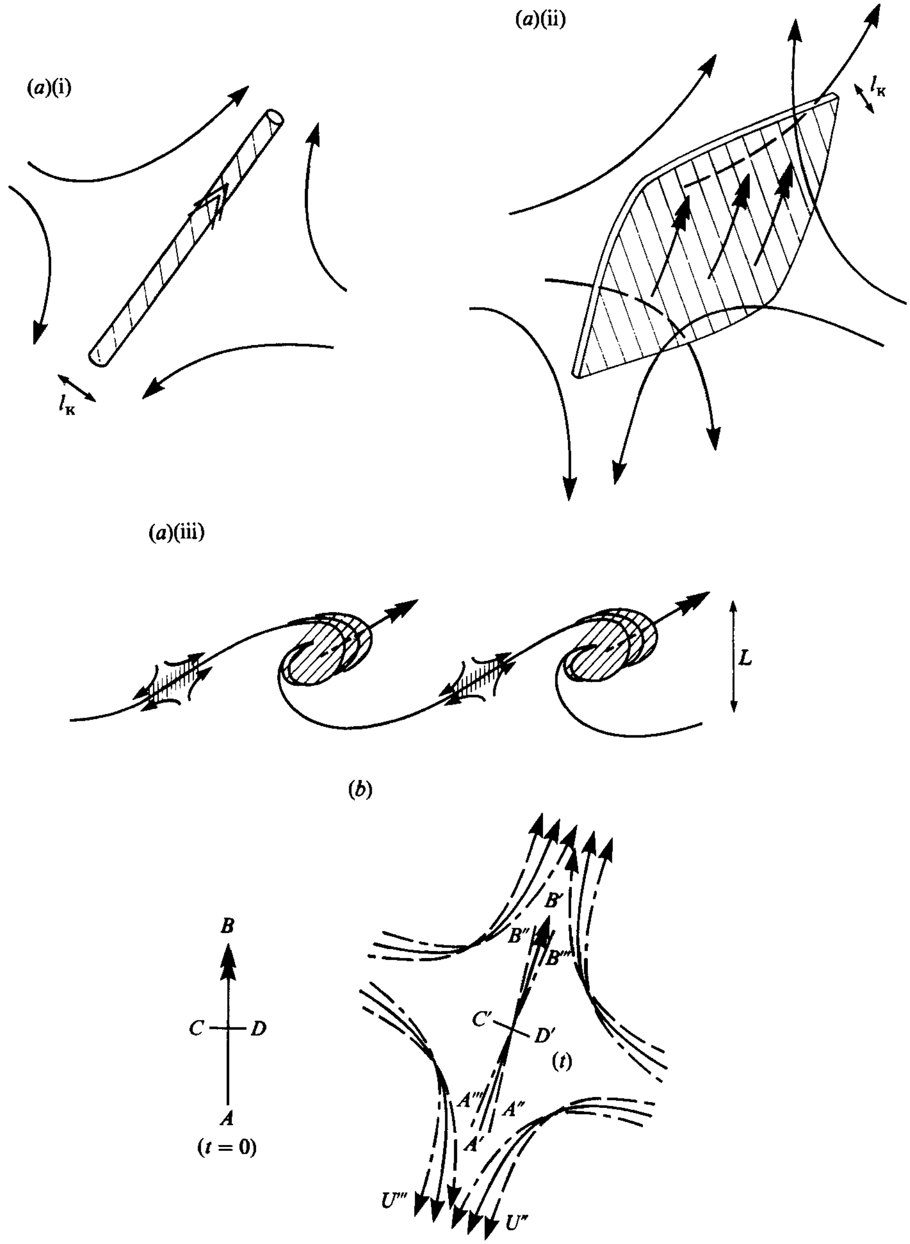

FIGURE 2. Sketches to illustrate vorticity dynamics. (*a*) (i), (ii) Vortex tubes and sheets on a scale of the Kolmogorov length surrounded by straining motions – these are narrow regions of high dissipation and high intermittent vorticity, and (iii) a region on much larger scale, *L*, where the vorticity is distributed and dissipation is weak – such as in parts of coherent structures. Regions of vorticity are indicated by //// and the direction by ——». (See Kerr 1985.) (*b*) Vortex elements of small-scale turbulence distorted in the period 0 to *t* by mean straining ——→ and by large eddies ——→ and ——→. The mean strain leads *AB* to change to *A′B′*; with the large eddies it changes to *A″B″* or *A‴B‴*. Note how that leads to a component of vorticity in the direction *C′D′* – an example of how rotational strain by large eddies reduces the anisotropy caused by the mean straining.

and the range of applicability can be clearly stated. For the general turbulence models mentioned earlier, the approximation and the range of validity are not well defined. Rapid distortion theory has been found to be a useful limiting case against which to test second-order closure theories where the turbulence undergoes mean distortion, or in regions close to surfaces; see for example Launder *et al.* (1975), Jeandel, Brison & Mathieu (1978) or Bertoglio (1986). But these comparisons have been restricted to the linear terms and have not in fact tested the closure approximation of representing third-order moments in terms of second-order moments.

The analysis presented here can be used to explore this key element in modelling turbulence and develop alternative models.

There is much similarity between this analysis of the dynamical interaction of large- and small-scale turbulent velocity fields and the interaction of large- and small-scale passive scalar fields, such as temperature, or passive vector fields, such as magnetic fields, in turbulent velocity fields (Moffatt 1981).

2. General assumptions and procedure

2.1. *Definitions, assumptions and averaging procedures*

We consider the development of a turbulent velocity field consisting of large-scale and small-scale turbulence, which are assumed to be statistically independent. The small-scale turbulence is advected by the large-scale turbulence in a frame of reference moving with the large-scale turbulence. We denote the lengthscales by L and l and the r.m.s. velocity scales by U' and u', respectively. As mentioned in the Introduction, we assume formally that

$$L \gg l, \tag{2.1}$$

$$\frac{U'}{L} \gg \frac{u'}{l}, \tag{2.2}$$

and that the time for the distortion t_D is of order (l/u').

Let $\boldsymbol{U}$ and $\boldsymbol{\Omega}$ be averages of the velocity and vorticity over a volume of lengthscale L^*, where $L \gg L^* \gg l$. The velocity and the vorticity are then written in terms of their large- and small-scale components, where both are defined in terms of fixed axes (or axes moving with the mean flow):

$$\boldsymbol{U}^* = \boldsymbol{U}+\boldsymbol{u}, \tag{2.3}$$

$$\boldsymbol{\Omega}^* = \boldsymbol{\Omega}+\boldsymbol{\omega}. \tag{2.4}$$

Of course, averages of $\boldsymbol{u}$ and $\boldsymbol{\omega}$ over a volume of scale L^* must vanish from the definition. The characteristic integral lengthscales of $\boldsymbol{U}$ and $\boldsymbol{\Omega}$ are $O(L)$, and those of $\boldsymbol{u}$ and $\boldsymbol{\omega}$ are $O(l)$. Therefore $\boldsymbol{U}$ and $\boldsymbol{\Omega}$ represent the velocity and vorticity of the large-scale turbulence, and $\boldsymbol{u}$ and $\boldsymbol{\omega}$ the small-scale ones (figure 3).

In studying the interactions between $\boldsymbol{U}$ and $\boldsymbol{u}$ we first calculate the development of the statistical properties of $\boldsymbol{u}$ in terms of its initial statistical description at time $t = 0$. The statistical properties of $\boldsymbol{u}$ are defined by averaging over a scale L^* in the vicinity of a point $\boldsymbol{X}^*(t)$ which moves at a velocity $\boldsymbol{U}(\boldsymbol{X}^*, t)$, and are calculated for each realization of $\boldsymbol{U}(\boldsymbol{x}^*, t)$ in the locality of $\boldsymbol{X}^*(t)$. By the ergodic hypothesis, these

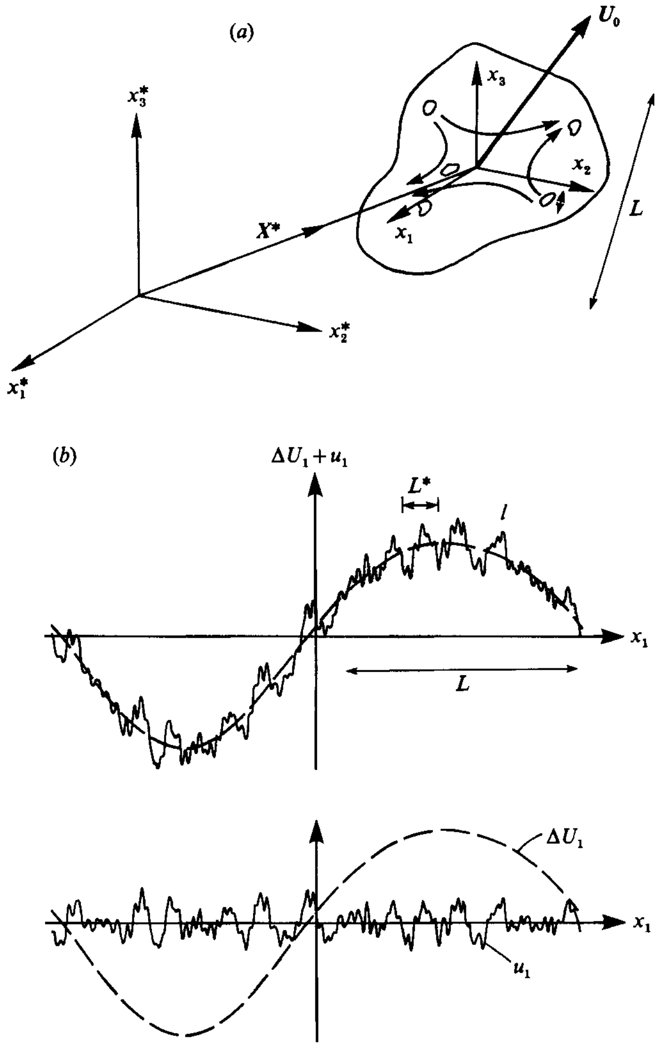

FIGURE 3. (a) A large eddy moving with an average velocity U_0 relative to the mean flow; the streamlines ⟶ indicate the straining flow in the large eddy $\Delta \boldsymbol{U}$; x_1, x_2, x_3 are axes moving with U_0; x_1^*, x_2^*, x_3^* are fixed axes (or fixed relative to any mean flow). The centre of the eddy is at $\boldsymbol{X}^*$. (b) The velocity recorded by an instrument making a rapid traverse across the large eddy (or by a mean flow advecting the eddy past a fixed probe). Note how the velocity is divided into large- and small-scale motions $(\Delta U_1, u_1)$.

statistics of $\boldsymbol{u}$ can also be calculated by taking an ensemble average of many identical realizations of $\boldsymbol{U}$. We shall denote these averages for given $\boldsymbol{U}$ by an overbar, so that a second moment of $\boldsymbol{u}$ (or of its Fourier transform) is denoted by

$$\overline{\boldsymbol{uu}}^{(U^{(i)})}$$

for the ith realization. This will usually be shortened to $\overline{\boldsymbol{uu}}$.

In the second stage of our calculation we consider an ensemble of realizations of $\boldsymbol{U}(\boldsymbol{x}^*, t)$, and take the average over this ensemble (denoted by $\langle\ \rangle$) of any particular statistical property of $\boldsymbol{u}$, which will have been calculated in the first stage for each realization of U. For example, the mean second moment averaged over all realizations of U becomes

$$\lim_{N\to\infty}\left(\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{uu}^{(U^{(i)})}\right)$$

which will be shortened to $\langle\boldsymbol{uu}\rangle$.

2.2. *Rapid distortion of the small scale by the large scale*

The condition (2.2) is a sufficient condition for the nonlinear terms $(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{\omega}$ and $(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{u}$ to be neglected in the equation for the vorticity $\boldsymbol{\omega}$ for an incompressible viscous flow with no boundary layer, and we have

$$\frac{\partial}{\partial t}\boldsymbol{\omega}+(\boldsymbol{U}\cdot\boldsymbol{\nabla})\boldsymbol{\omega}+(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{\Omega}=(\boldsymbol{\Omega}\cdot\boldsymbol{\nabla})\boldsymbol{u}+(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{U}+\nu\nabla^2\boldsymbol{\omega}, \tag{2.5}$$

where ν is the kinematic viscosity of the fluid. Sometimes ν is replaced by an eddy viscosity as a simple representation of the nonlinear effect of scales smaller than l (Townsend 1976). By not representing this process, we restrict the value of t for which the calculations are valid.

Since the turbulence scale of $\boldsymbol{u}$ is so much less than that of $\boldsymbol{U}$ and uncorrelated with it, and since $\boldsymbol{u}$ is to be calculated within a distance of order L^* from a reference point $\boldsymbol{X}^*$, moving with the velocity $\boldsymbol{U}(\boldsymbol{X}^*,t)$, $\boldsymbol{U}$ can be expressed as a sum of translational velocity $\boldsymbol{U}_0$ and a uniform straining motion $\Delta\boldsymbol{U}$.

Then $\Delta\boldsymbol{U}$ and the small-scale motion can be defined in terms of axes $\boldsymbol{x}(=(x,y,z))$ moving with the velocity $\boldsymbol{U}_0$, viz.

$$\boldsymbol{U}^*=\boldsymbol{U}_0+\Delta\boldsymbol{U}(\boldsymbol{x},t)+\boldsymbol{u}(\boldsymbol{x},t), \tag{2.6a}$$

where

$$\Delta\boldsymbol{U}=\boldsymbol{\alpha}\cdot\boldsymbol{x}\quad\text{or}\quad\Delta U_i=\alpha_{ij}x_j, \tag{2.6b}$$

$$\boldsymbol{x}=\boldsymbol{x}^*-\boldsymbol{X}^* \tag{2.6c}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{X}^*=\boldsymbol{U}_0(\boldsymbol{X}^*,t). \tag{2.6d}$$

Note that the orientation of the vector $\boldsymbol{u}$ is not dependent on the direction of $\boldsymbol{U}_0$. The vorticity is also advected with the velocity $\boldsymbol{U}_0$, so

$$\Omega_i(\boldsymbol{x},t)=\epsilon_{ijk}\alpha_{kj}\quad\text{and}\quad\boldsymbol{\omega}=\boldsymbol{\omega}(\boldsymbol{x},t), \tag{2.7}$$

where ϵ_{ijk} is the alternating tensor.

For incompressible large-scale motion, since $\boldsymbol{\nabla}\cdot\Delta\boldsymbol{U}=0$,

$$\alpha_{ll}=0. \tag{2.8}$$

(The summation convention is used for repeated suffices unless otherwise stated.)

In the subsequent analysis $\boldsymbol{\alpha}$ is assumed to be constant in time, but this restriction can be removed.

Substituting the forms for $\boldsymbol{U}^*$ and $\boldsymbol{\Omega}^*$ given by (2.6) and (2.7) into the equation for small-scale vorticity (2.5), we get the governing equation for $\boldsymbol{u}$ and $\boldsymbol{\omega}$:

$$\frac{\partial}{\partial t}\omega_i+\alpha_{jk}x_k\frac{\partial}{\partial x_j}\omega_i=\epsilon_{jkl}\alpha_{lk}\frac{\partial}{\partial x_j}u_i+\omega_j\alpha_{ij}+\nu\nabla^2\omega_i. \tag{2.9}$$

Since this is a linear equation, a general solution can be constructed from solutions for each Fourier component in $\boldsymbol{u}$ and $\boldsymbol{\omega}$. Following the method of Moffatt (1967) and Townsend (1976), we write

$$u_i(\boldsymbol{x},t) = \int \hat{u}_i(\boldsymbol{\chi},t)\exp[i\boldsymbol{\chi}(t)\cdot\boldsymbol{x}]\,\mathrm{d}\boldsymbol{\chi}, \tag{2.10}$$

$$\omega_i(\boldsymbol{x},t) = \int \hat{\omega}_i(\boldsymbol{\chi},t)\exp[\mathrm{i}\boldsymbol{\chi}(t)\cdot\boldsymbol{x}]\,\mathrm{d}\boldsymbol{\chi}, \tag{2.11}$$

where

$$\hat{u}_i(\boldsymbol{\chi},t) = \mathrm{i}\epsilon_{ijk}\frac{\chi_j}{\chi^2}\hat{\omega}_k(\boldsymbol{\chi},t). \tag{2.12}$$

Substituting (2.10)–(2.12) into (2.9) and allowing the wavenumber $\boldsymbol{\chi}(t)$ to change in time appropriately, as well as the amplitudes $\hat{\boldsymbol{u}}(\boldsymbol{\chi},t), \hat{\boldsymbol{\omega}}(\boldsymbol{\chi},t)$, the advective term on the left-hand side of (2.9) which is linear in $\boldsymbol{x}$ is removed. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\chi_i = -\chi_j\alpha_{ji}, \tag{2.13}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\omega}_i = \beta_{ij}\hat{\omega}_j, \tag{2.14}$$

where

$$\beta_{ij}(t) = \alpha_{ji} + \frac{\chi_i\chi_k}{\chi^2}(\alpha_{kj}-\alpha_{jk}) - \nu\chi^2\delta_{ij}. \tag{2.15a}$$

Note that for irrotational inviscid distortion

$$\alpha_{kj} = \alpha_{jk} \quad \text{and therefore} \quad \beta_{ij} = \alpha_{ji}. \tag{2.15b}$$

For rotational distortions β_{ij} is a nonlinear function of α_{ij}, because χ depends on α_{ij}. In that case β_{ij} depends on the history of α_{ij}. Townsend (1980) has calculated $\hat{u}_i(\boldsymbol{\chi},t)$ for a number of different combinations of α_{ij}, to show how changing the sequence of rotational and irrotational distortions affects the change of $\boldsymbol{u}$ and $\boldsymbol{\omega}$.

The unique solutions to (2.13) and (2.14) can be expressed in terms of the values of $\boldsymbol{\chi}$ and $\hat{\boldsymbol{\omega}}$ by a deformation tensor $\tilde{\boldsymbol{S}}$ as

$$\boldsymbol{\chi} = \boldsymbol{\kappa}\cdot\tilde{\boldsymbol{S}} \tag{2.16}$$

and

$$\hat{\boldsymbol{\omega}}(\boldsymbol{\chi},t) = \boldsymbol{T}\cdot\hat{\boldsymbol{\omega}}(\boldsymbol{\kappa},0), \tag{2.17}$$

where $\boldsymbol{\kappa} = \boldsymbol{\chi}(0)$. Here $\tilde{\boldsymbol{S}}$ is the inverse of the matrix $\boldsymbol{S}$, defined as

$$\boldsymbol{S} = \sum_{n=0}^{\infty}\int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\ldots\int_0^{t_{n-1}}\mathrm{d}t_n\,\boldsymbol{\alpha}(t_1)\cdot\boldsymbol{\alpha}(t_2)\ldots\boldsymbol{\alpha}(t_n)$$
$$= \exp\left[\int_0^t \boldsymbol{\alpha}\,\mathrm{d}t'\right]. \tag{2.18a}$$

For a constant strain rate

$$\boldsymbol{S} = \exp[\boldsymbol{\alpha}t]$$
$$= \sum_{n=0}^{\infty}\frac{1}{n!}\boldsymbol{\alpha}^n t^n, \tag{2.18b}$$

where the notation

$$(\boldsymbol{\alpha}^n)_{ij} = \alpha_{ia}\alpha_{ab}\alpha_{bc}\ldots\alpha_{kj} \tag{2.18c}$$

denotes a matrix operation. The vorticity deformation matrix is given by

$$\boldsymbol{T} = \exp\left[\int_0^t \boldsymbol{\beta}(t')\,\mathrm{d}t'\right], \tag{2.19a}$$

using the same notation for the exponential as in (2.18). Note that for an irrotational, inviscid distortion, since $\boldsymbol{\alpha} = \boldsymbol{\beta}$,

$$\boldsymbol{S} = \boldsymbol{T} \tag{2.19b}$$

as pointed out by Durbin (1981). For rotational disturbances where $\boldsymbol{S}$ is independent of $\boldsymbol{\chi}$ and $\boldsymbol{T}$ is dependent on $\boldsymbol{\alpha}$ and $\boldsymbol{\chi}$, these two matrices are not equal.

Spatial spectra of the velocity and vorticity for each realization of the homogeneous small-scale turbulence are defined in terms of the wavenumbers $\boldsymbol{\chi}$ defined at time t as follows:

$$\Phi_{ij}(\boldsymbol{\chi}, t) = \overline{\hat{u}_i(\boldsymbol{\chi}', t)\, \hat{u}_i^{\dagger}(\boldsymbol{\chi}, t)}\, \delta(\boldsymbol{\chi}' - \boldsymbol{\chi}) \tag{2.20}$$

and

$$\Omega_{ij}(\boldsymbol{\chi}, t) = \overline{\hat{\omega}_i(\boldsymbol{\chi}', t)\, \hat{\omega}_j^{\dagger}(\boldsymbol{\chi}, t)}\, \delta(\boldsymbol{\chi}' - \boldsymbol{\chi}), \tag{2.21}$$

where the dagger denotes the complex conjugate and the overbar an ensemble average for the small-scale turbulence. (This could be a spatial average over a scale L^*.)

It follows from (2.12) that these spectra are related to each other by

$$\boldsymbol{\Phi}(\boldsymbol{\chi}, t) = \frac{1}{\chi^2} \boldsymbol{D}(\boldsymbol{\chi}) : \boldsymbol{\Omega}(\boldsymbol{\chi}, t) \tag{2.22}$$

and, by inversion

$$\boldsymbol{\Omega}(\boldsymbol{\chi}, t) = \chi^2 \boldsymbol{D}(\boldsymbol{\chi}) : \boldsymbol{\Phi}(\boldsymbol{\chi}, t), \tag{2.23}$$

where the fourth-order tensor

$$D_{ab,cd}(\chi) = \left(\delta_{ab} - \frac{\chi_a \chi_b}{\chi^2}\right) \delta_{cd} - \delta_{ac}\delta_{bd}. \tag{2.24}$$

Combination of (2.17) and (2.21) gives us the time-development of $\boldsymbol{\Omega}$ in terms of its initial value as

$$\boldsymbol{\Omega}(\boldsymbol{\chi}, t) = \boldsymbol{T}(t) \cdot \boldsymbol{\Omega}(\boldsymbol{\kappa}, t_0) \cdot \boldsymbol{T}^{\mathrm{T}}(t), \tag{2.25}$$

where $\boldsymbol{T}^{\mathrm{T}}$ is the transposed matrix of $\boldsymbol{T}$. We have used the incompressibility condition

$$\mathrm{d}\boldsymbol{\chi} = \mathrm{d}\boldsymbol{\kappa} \quad \text{or} \quad \delta(\boldsymbol{\chi} - \boldsymbol{\chi}') = \delta(\boldsymbol{\kappa} - \boldsymbol{\kappa}'). \tag{2.26}$$

As for the energy-spectrum tensor, we find, using (2.22), (2.23) and (2.25), that

$$\boldsymbol{\Phi}(\boldsymbol{\chi}, t) = \left(\frac{\kappa}{\chi}\right)^2 \boldsymbol{D}(\boldsymbol{\chi}) : \{\boldsymbol{T}(t) \cdot [\boldsymbol{D}(\boldsymbol{\kappa}) : \boldsymbol{\Phi}(\boldsymbol{\kappa}, t_0)] \cdot \boldsymbol{T}^{\mathrm{T}}(t)\}. \tag{2.27}$$

Alternatively this can be expressed in tensor form as

$$\Phi_{ij}(\boldsymbol{\chi}, t) = M_{ij,rs}(\boldsymbol{\kappa}, t)\, \Phi_{rs}(\boldsymbol{\kappa}, t_0), \tag{2.28a}$$

where the tensor

$$M_{ij,rs}(\boldsymbol{\kappa}, t) = \left(\frac{\kappa}{\chi}\right)^2 D_{ij,gh}(\chi)\, T_{gm}(t)\, T_{hn}(t)\, D_{mn,rs}(\boldsymbol{\kappa}) \tag{2.28b}$$

is a transfer function which is a functional of the particular realization of the large-scale velocity field ΔU. It does not depend on the translational velocity U_0 or on any property of the small-scale $\boldsymbol{u}$.

From (2.27) or (2.28) the second moments of $\boldsymbol{u}$ can be obtained by integration either of the initial energy-spectrum tensor $\Phi_{rs}(\boldsymbol{\kappa}, 0)$ or of the initial cross-correlation $\overline{u_r(\boldsymbol{x}, 0)\, u_s(\boldsymbol{x}+\boldsymbol{y}, 0)}$:

$$\overline{u_i u_j} = \int M_{ij,rs}(\boldsymbol{\kappa}, t)\, \Phi_{rs}(\boldsymbol{\kappa}, 0)\, \mathrm{d}\boldsymbol{\kappa} \tag{2.29a}$$

$$= \iint M_{ij,rs}(\boldsymbol{\kappa}, t)\, \overline{u_r(\boldsymbol{x}, 0)\, u_s(\boldsymbol{x}+\boldsymbol{y}, 0)} \exp[\mathrm{i}\boldsymbol{\kappa} \cdot \boldsymbol{y}]\, \mathrm{d}\boldsymbol{y}\, \mathrm{d}\boldsymbol{\kappa}. \tag{2.29b}$$

In general this linear rapid distortion theory enables the nth moment of any component or components of the velocity field to be expressed schematically at time t in terms of nth moments of the velocity field at time 0 as

$$\overline{\{\boldsymbol{u}\}^{(n)}}(t) = M^{(n)}(t) * \overline{\{\boldsymbol{u}\}^{(n)}}(0), \tag{2.30}$$

where $M^{(n)}(t)$ is a linear integral operator which is a function of time, and a functional of the rate-of-strain tensor $\boldsymbol{\alpha}$.

2.3. *Averaging over an ensemble of large-scale motions*

We have shown in §2.2 that the calculation of second-order moments $\overline{\boldsymbol{uu}}$ such as $\Phi_{ij}(\chi, t)$, $\Omega_{ij}(\chi, t)$ or $\overline{u_i u_j}(t)$, for a given realization of the large-scale velocity field $\boldsymbol{U}$ in the vicinity of the moving reference point $\boldsymbol{X}^*(t)$, only depends on the particular value of $\partial U_i/\partial x_j$ or $\boldsymbol{\alpha}$ within a distance L^* of $\boldsymbol{X}^*$ in that realization. It does not depend explicitly on $\boldsymbol{U}(\boldsymbol{X}^*, t)$.

Consequently in calculating the ensemble average of any statistical property of $\boldsymbol{u}$, e.g. $\overline{\boldsymbol{uu}}$, over all realizations of $\boldsymbol{U}$, we need only consider the ensemble of realizations of the rate of strain $\boldsymbol{\alpha}$ or $\partial U_i/\partial x_j$. We could denote this ensemble by $\boldsymbol{\alpha}^{(k)}, k = 1, 2, \ldots, N$. Thus for a second-order moment, this ensemble is

$$\langle \overline{\boldsymbol{uu}} \rangle = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} (\overline{\boldsymbol{uu}}^{\alpha^{(k)}}). \tag{2.31}$$

Inspection of (2.28), (2.29) and (2.30) shows how this ensemble average is to be evaluated. We note that, since $\boldsymbol{\Phi}(\boldsymbol{\kappa}, 0)$ or $\overline{\boldsymbol{u}(\boldsymbol{x}, 0)\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{y}, 0)}$ or $\{\boldsymbol{u}\}^{(n)}(0)$ are all independent of $\boldsymbol{\alpha}^{(k)}$, the calculation of the ensemble average over all values of $\boldsymbol{\alpha}^{(k)}$ only applies to the transfer-function tensor, i.e. $M_{ij,rs}$ for second moments and $M^{(n)}(t)$ for nth moments. Thus

$$\langle \Phi_{ij}(\chi, t) \rangle_{\chi\,\text{fixed}} = \langle M_{ij,rs}(\boldsymbol{\kappa}, t) \rangle\, \Phi_{rs}(\boldsymbol{\kappa}, 0)_{\chi\,\text{fixed}} \tag{2.32}$$

or

$$\langle \overline{u_i u_j}(t) \rangle = \iint \langle M_{ij,rs}(\boldsymbol{\kappa}, t) \rangle\, \overline{u_r(\boldsymbol{x}, 0)\, u_s(\boldsymbol{x}+\boldsymbol{y}, 0)} \exp[\mathrm{i}\boldsymbol{\kappa} \cdot \boldsymbol{y}]\, \mathrm{d}\boldsymbol{y}\, \mathrm{d}\boldsymbol{\kappa} \tag{2.33}$$

or in general

$$\overline{\{\boldsymbol{u}\}^{(n)}}(t) = \langle M^{(n)}(t) \rangle * \overline{\{\boldsymbol{u}\}^{(n)}}(0). \tag{2.34}$$

There are two important implications of the independence of the averaging process, the first being that the calculations of the ensemble average of $M_{ij,rs}$ need not be performed for each initial nth moment distribution, so for example once $\langle M_{ij,rs} \rangle$ has been calculated, $\langle \overline{u_i u_j}(t) \rangle$ can be calculated for many different forms of $\Phi_{rs}(\boldsymbol{\kappa}, 0)$. But $\langle \Phi_{ij}(\chi, t) \rangle$ or $\langle \overline{u_i(\boldsymbol{x})\, u_j(\boldsymbol{x}+\boldsymbol{y})} \rangle$ can only be calculated by averaging over $M_{ij,rs}$ and Φ_{rs} for each form of Φ_{rs} as in (2.32).

The second implication is that the averaging operations can be carried out at the same time or at different times. Thus, for example, in a stationary process, the small-scale averaging and the large-scale averaging $\langle\ \rangle$ can both be evaluated by integration with respect to time. This means that the change in, say, $\langle \overline{u_i u_j} \rangle$ can be measured by a probe that is fixed relative to the mean flow or fixed in space with the mean flow advecting the turbulence past it. This is the procedure in a typical laboratory experiment, where small scales are deliberately introduced (Tan-atichat *et al.* 1982), or where the turbulence is analysed in terms of large-scale structures and small-scale turbulence (e.g. Hussain 1983). It also means that it is not necessary to make measurements in a frame moving with the large-scale flow $\boldsymbol{U}$ to test the predictions for the quantities averaged over an ensemble of realizations of $\boldsymbol{U}$ and $\boldsymbol{u}$, e.g. $\langle\overline{u_i u_j}\rangle$. But it is necessary to make such 'Lagrangian' measurements to test the predictions for changes in the small-scale $\boldsymbol{u}$-quantities which are averaged over L^* or over an ensemble of realizations for a particular realization of the large-scale flow field $\boldsymbol{U}$ (e.g. the kth realization $\boldsymbol{U}^{(k)}$). (However, if the mean velocity U is large enough relative to $\boldsymbol{U}'$ and $\boldsymbol{u}'$, the evolution of the small scale, e.g. $\overline{u_i u_j}$, can be calculated from measurements of the velocity field over a time of order L^*/U first at x_1^* and then at another point at time t later $(x_1^* + Ut)$ (figure 3*a*). With multiple arrays of probes it should be possible to define different realizations of the large-scale velocity field and thence test the connection between $\overline{u_i u_j}^{\alpha^{(k)}}$ and $\boldsymbol{U}^{(k)}$.)

2.4. *Analysis by means of second-order moment equations and pressure–velocity correlations*

Many previous analyses of the interaction of large- and small-scale motions in turbulence, especially the interaction of waves and small-scale turbulence (e.g. Finnigan & Einaudi 1981), have been based on the equations of second-order moments. In the notation of this paper, and ignoring third-order moments and dissipative effects, the equations for the ensemble mean of the second-order moments of the locally homogeneous small-scale turbulence are

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\overline{u_i u_j}\rangle = \langle P_{\mathrm{RS}}\rangle_{ij} + \langle P_{\mathrm{PG}}\rangle_{ij}, \tag{2.35a}$$

where the term for the production of Reynolds stress by the Reynolds stresses themselves, $\langle P_{\mathrm{RS}}\rangle$, is

$$\begin{aligned}\langle P_{\mathrm{RS}}\rangle_{ij} &= -\left\langle \overline{u_i u_k}\frac{\partial U_j}{\partial x_k} + \overline{u_j u_k}\frac{\partial U_i}{\partial x_k}\right\rangle \\ &= -\left\langle \int (\alpha_{jk}\,\Phi_{ik}(\boldsymbol{\chi}) + \alpha_{ik}\,\Phi_{jk}(\boldsymbol{\chi}))\,\mathrm{d}\boldsymbol{\chi}\right\rangle,\end{aligned} \tag{2.35b}$$

and the term for the production sharing of Reynolds stresses in different orientations by the pressure gradient, $\langle P_{\mathrm{PG}}\rangle$, is

$$\langle P_{\mathrm{PG}}\rangle_{ij} = -\frac{1}{\rho}\left\langle \overline{u_j\frac{\partial p}{\partial x_i}} + \overline{u_i\frac{\partial p}{\partial x_j}}\right\rangle \tag{2.35c}$$

and p satisfies

$$\frac{1}{\rho}\nabla^2 p = -2\frac{\partial U_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} = -2\alpha_{ij}\frac{\partial u_j}{\partial x_i}. \tag{2.36}$$

Note that the terms like $\overline{u_i u_k}$ in P_{RS} and $\partial p/\partial x_j$ in P_{PG} are functions of $\boldsymbol{\alpha}$ and therefore the large-scale ensemble average is taken over the product $\overline{u_i u_k}\,\partial U_j/\partial x_k$ and $\overline{u_j\,\partial p/\partial x_j}$.

The term P_{PG} can be expressed by standard methods in terms of the local velocity energy spectrum tensor $\Phi_{ij}(\boldsymbol{\chi}, t)$ as

$$\langle P_{\mathrm{PG}}\rangle_{ij} = 2\left\langle \int \frac{\chi_a\,\alpha_{ab}}{\chi^2}(\chi_i\,\Phi_{bj}(\boldsymbol{\chi}) + \chi_j\,\Phi_{bi}(\boldsymbol{\chi}))\,\mathrm{d}\boldsymbol{\chi}\right\rangle. \tag{2.37}$$

In general, there is insufficient information in (2.35) and (2.36) to enable $\langle\overline{u_i u_j}\rangle$ to be calculated, although tensorial invariance can be used to infer the form of P_{PG}. However, by using the rapid distortion analysis described in §2.2 and the procedure for taking ensemble averages over the large-scale straining motions, the term P_{PG} involving pressure gradient–velocity correlations can be calculated. Then $\langle\overline{u_i u_j}\rangle$ can be calculated from (2.35). In the framework of our assumptions this procedure is no better than using r.d.t. directly.

In many approximate calculations of turbulent flows, it is generally assumed that effects of pressure gradients on the anisotropy of turbulence are directly proportional to the anisotropy of the second-order moments of the turbulence (e.g. Launder *et al.* 1975; Lumley 1978), because $\langle P_{\mathrm{PG}} \rangle = 0$ if the large-scale and the small-scale turbulence are isotropic. This is an assumption we shall be able to consider in §4, after computing (2.37) by r.d.t.

3. Isotropic irrotational large-scale straining

3.1. *Arbitrary strains*

We first examine the effects on the small-scale turbulence of irrotational straining of constant type by the large-scale turbulence. We assume that the Reynolds number is large enough to ignore viscous effects. In this case $\boldsymbol{\alpha}$ is symmetric and

$$\alpha_{ij} = \alpha_{ji}. \tag{3.1}$$

Let us introduce Eulerian angles θ, ϕ, ψ which describe the directions of the three principal axes x', y', z' of the strain (see figure 4). These angles run through the following range,

$$0 \leqslant \theta \leqslant \pi, \quad 0 \leqslant \phi < 2\pi, \quad 0 \leqslant \psi < 2\pi. \tag{3.2}$$

If we denote the principal values of $\boldsymbol{\alpha}$ in the directions of the x', y', z' axes by $\alpha_1, \alpha_2, \alpha_3$, respectively, then $\boldsymbol{\alpha}$ is expressed as

$$\boldsymbol{\alpha} = \tilde{\boldsymbol{A}} \cdot \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix} \cdot \boldsymbol{A}, \tag{3.3}$$

where

$$\alpha_1 + \alpha_2 + \alpha_3 = 0, \tag{3.4}$$

$$\boldsymbol{A} = \begin{pmatrix} \cos\phi\cos\psi - \cos\theta\sin\phi\sin\psi & \sin\phi\cos\psi + \cos\theta\cos\phi\sin\psi & \sin\theta\sin\psi \\ -\cos\phi\sin\psi - \cos\theta\sin\phi\cos\psi & -\sin\phi\sin\psi + \cos\theta\cos\phi\cos\psi & \sin\theta\cos\psi \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}. \tag{3.5}$$

Combination of (2.18*b*, *c*) and (3.3) gives the changes in the wavenumber matrix $\boldsymbol{S}$ after n operations

$$\boldsymbol{S}^n = \tilde{\boldsymbol{A}} \cdot \begin{pmatrix} \exp[n\alpha_1 t] & 0 & 0 \\ 0 & \exp[n\alpha_2 t] & 0 \\ 0 & 0 & \exp[n\alpha_3 t] \end{pmatrix} \cdot \boldsymbol{A}. \tag{3.6}$$

Note that $\boldsymbol{S}$ is also a symmetrix matrix.

Recalling the results of (2.15*b*) and (2.19*b*) for the vorticity distortion matrix $\boldsymbol{T}$ in inviscid irrotational strains, we have

$$\boldsymbol{\beta} = \boldsymbol{\alpha}^{\mathrm{T}} (= \boldsymbol{\alpha}) \tag{3.7}$$

and

$$\boldsymbol{T} = \boldsymbol{S}, \tag{3.8}$$

i.e. $\boldsymbol{T}$ is determined only by the straining tensor $\boldsymbol{\alpha}$ and does not depend upon a particular wavenumber $\boldsymbol{\chi}(t)$.

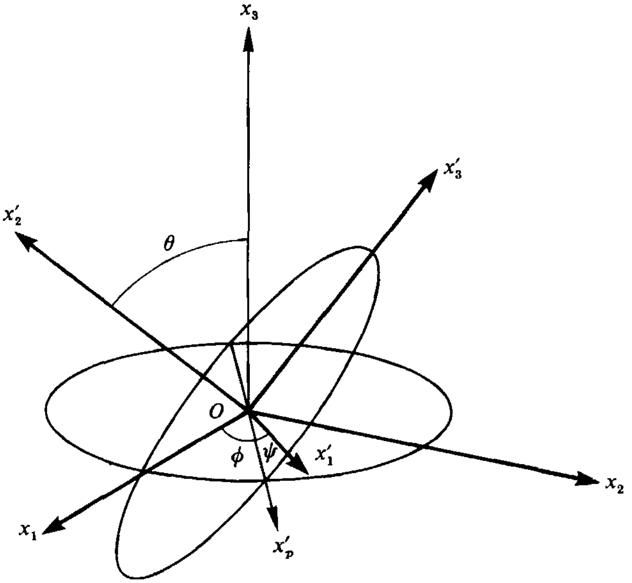

FIGURE 4. The Eulerian angles ϕ, ψ, θ between the axes (x_1, x_2, x_3) with fixed orientation and the axes (x_1', x_2', x_3') parallel to the principal axes of the rate-of-strain tensor α_{ij} of the large-scale velocity field U. Note that the angles ϕ and ψ are defined relative to Ox_p', the projection of Ox_1' onto the plane $Ox_1 x_2$.

Note that by integrating both sides of (2.25) with respect to χ, we obtain a simple relationship connecting the vorticity moments before and after a distortion:

$$\boldsymbol{W}(t) = \boldsymbol{S}(t) \cdot \boldsymbol{W}(0) \cdot \boldsymbol{S}^{\mathrm{T}}(t), \tag{3.9}$$

where

$$W_{ij}(t) = \overline{\omega_i(\boldsymbol{x}, t)\, \omega_j(\boldsymbol{x}, t)} = \int \Omega_{ij}(\boldsymbol{\chi}, t)\, \mathrm{d}\boldsymbol{\chi} \tag{3.10}$$

is the vorticity moment.

Now we proceed to consider an average effect of many strains with different strengths and orientations. Cases where the large-scale motions are characteristic of large-scale isotropic turbulence are of interest, so that the direction of strains will be supposed to be distributed isotropically in space. The probability density of this distribution of θ, ϕ, ψ is then given by

$$P_{\theta\phi\psi}(\theta, \phi, \psi) = \frac{1}{8\pi^2} \sin\theta. \tag{3.11}$$

When the distortion is small, i.e. $\|\boldsymbol{\alpha}\|\, t \ll 1$, where $\|\boldsymbol{\alpha}\|$ is the Euclidean norm of $\boldsymbol{\alpha}$, it is not necessary to specify the probability distribution of $\alpha_1, \alpha_2, \alpha_3$ to obtain an ensemble average; only the variances of second moments may be necessary. But when $\|\boldsymbol{\alpha}\|\, t \gtrsim 1$, all the moments of $\boldsymbol{\alpha}$ must be specified, which is to say the probability distribution. Stewart's (1951) measurements suggest that the probability distribution for the strength of large-scale strains is Gaussian, which, for incompressible flow, has the form

$$P_{\alpha_1\alpha_2\alpha_3}(\alpha_1, \alpha_2, \alpha_3) = \frac{1}{\sqrt{3\pi\alpha_0^2}} \exp\left[-\frac{\alpha_1^2+\alpha_2^2+\alpha_3^2}{3\alpha_0^2}\right] \delta(\alpha_1+\alpha_2+\alpha_3), \tag{3.12}$$

where α_0 is the standard deviation of α_i, i.e.

$$\alpha_0^2 = \iiint \alpha_i^2 P_{\alpha_1\alpha_2\alpha_3}(\alpha_1, \alpha_2, \alpha_3)\, \mathrm{d}\alpha_1\, \mathrm{d}\alpha_2\, \mathrm{d}\alpha_3 \quad (i = 1, 2, 3). \tag{3.13}$$

Also α_0 is the order of magnitude of the root mean square of the rate of strain of the large-scale motion.

The development of the ensemble-averaged vorticity spectrum tensor $\langle \boldsymbol{\Omega} \rangle$ is derived from (2.25) with $\boldsymbol{T} = \boldsymbol{S}$ by first averaging over the Eulerian angles, for fixed values of $\alpha_1, \alpha_2, \alpha_3$:

$$\langle \boldsymbol{\Omega}(\chi, t) \rangle_{\substack{\theta\phi\psi \\ \chi\,\text{fixed}}} = \langle \boldsymbol{S}(t) \cdot \boldsymbol{\Omega}(\boldsymbol{\kappa}, 0) \cdot \boldsymbol{S}^{\mathrm{T}}(t) \rangle_{\substack{\theta\phi\psi \\ \chi\,\text{fixed}}}. \tag{3.14}$$

Since $\boldsymbol{\kappa}$ is a function of χ and θ, ϕ, ψ, and the average must be carried out for fixed χ, this average cannot actually be done unless the initial form of $\boldsymbol{\Omega}$ is specified. In §5.2 we shall give an example of $\boldsymbol{\Omega}$ for isotropic large-scale turbulence, where we also average over $\alpha_1, \alpha_2, \alpha_3$ using (3.12).

Similarly, the evolution of the energy-spectrum tensor $\langle \boldsymbol{\Phi} \rangle$ can be derived from (2.27) by first averaging over θ, ϕ, ψ for given $\alpha_1, \alpha_2, \alpha_3$:

$$\langle \boldsymbol{\Phi}(\chi, t) \rangle_{\substack{\theta\phi\psi \\ \chi\,\text{fixed}}} = \frac{1}{\chi^2} \boldsymbol{D}(\chi) : \langle \kappa^2 \boldsymbol{S}(t) \cdot (\boldsymbol{D}(\boldsymbol{\kappa}) : \boldsymbol{\Phi}(\boldsymbol{\kappa}, 0)) \cdot \boldsymbol{S}^{\mathrm{T}}(t) \rangle_{\substack{\theta\phi\psi \\ \chi\,\text{fixed}}}. \tag{3.15}$$

3.2. *Ensemble-averaged vorticity moments*

Let us denote the ensemble average of the vorticity-moment tensor over θ, ϕ, ψ, for given $\alpha_1, \alpha_2, \alpha_3$, by

$$\langle \overline{\omega_i \omega_j}(t) \rangle_{\theta\phi\psi} = \langle W_{ij}(t) \rangle_{\theta\phi\psi} = \left\langle \int \Omega_{ij}(\chi, t)\, \mathrm{d}\chi \right\rangle_{\theta\phi\psi}, \tag{3.16}$$

and introduce a new variable which is a convenient measure of anisotropy of the vorticity field

$$\langle a_{ij}(t) \rangle_{\theta\phi\psi} = \frac{\langle W_{ij}(t) \rangle_{\theta\phi\psi}}{\langle W_{ll}(t) \rangle_{\theta\phi\psi}} - \tfrac{1}{3}\delta_{ij}. \tag{3.17}$$

Note that $\langle a_{ij}(t) \rangle_{\theta\phi\psi}$ vanishes identically for isotropic turbulence.

By integrating (3.14) with respect to wavenumber, using the incompressibility condition (2.26), we obtain the amplification of the mean-square vorticity over the ensemble of different orientations of distortion, for given $\alpha_1, \alpha_2, \alpha_3$:

$$\langle W_{ll}(t) \rangle_{\theta\phi\psi} = \tfrac{1}{3}\{\exp[2\alpha_1 t] + \exp[2\alpha_2 t] + \exp[2\alpha_3 t]\}\, W_{ll}(0). \tag{3.18}$$

Since the arithmetic mean is greater than the geometric mean, and since $\Sigma_{i=1}^3 \alpha_i = 0$ by incompressibility,

$$\frac{1}{3} \sum_{i=1}^{3} \exp[2\alpha_i t] \geqslant \left\{ \prod_{i=1}^{3} \exp[2\alpha_i t] \right\}^{\frac{1}{3}} = \exp\left[\tfrac{2}{3} t \sum_{i=1}^{3} \alpha_i \right] = 1.$$

Consequently the mean-square vorticity or the enstrophy must increase or

$$\langle \overline{\omega^2}(t) \rangle_{\theta\phi\psi} \geqslant \langle \overline{\omega^2}(0) \rangle_{\theta\phi\psi}. \tag{3.19}$$

The vorticity anisotropy tensor develops as

$$\langle a_{ij}(t) \rangle_{\theta\phi\psi} = F_\omega(t) \langle a_{ij}(0) \rangle_{\theta\phi\psi}, \tag{3.20}$$

where

$$F_\omega(t) = \frac{2}{5} + \frac{3}{5} \frac{\exp[-\alpha_1 t] + \exp[-\alpha_2 t] + \exp[-\alpha_3 t]}{\exp[2\alpha_1 t] + \exp[2\alpha_2 t] + \exp[2\alpha_3 t]}. \tag{3.21}$$

Since $\alpha_3 = -(\alpha_1+\alpha_2)$,

$$F_\omega(t) = \frac{2}{5}+\frac{3}{5}\frac{\exp[-t']+\exp[-\lambda t']+\exp[(1+\lambda)t']}{\exp[2t']+\exp[2\lambda t']+\exp[-2(1+\lambda)t']},$$

where $t' = \alpha_1 t$ and $\lambda = \alpha_2/\alpha_1$, thence

$$\tfrac{2}{5} \leqslant F_\omega(t) \leqslant 1. \tag{3.22}$$

Therefore (3.20) implies that anisotropy is always decreased by an ensemble of irrotational isotropic straining flows. However, the anisotropy cannot be reduced to zero. From (3.22) we see that the anisotropy cannot be less than $\frac{2}{5}$ of its original value.

The understanding of this surprising result may be somewhat helped by considering two vortex lines with different lengths l_1 and l_2 on the x- and y-axes. The same straining is applied at two orthogonal orientations, i.e.

$$\text{first}\quad \boldsymbol{\alpha} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \quad\text{and then}\quad \boldsymbol{\alpha} = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}.$$

The results are averaged leading to an increase in both l_1 and l_2 (because

$$\tfrac{1}{2}(e^t+\mathrm{e}^{-t}) \approx \tfrac{1}{2}\mathrm{e}^t$$

when t is large), but the ratio l_1/l_2 is equal to its initial value for large time. If the irrotational straining term is applied at all the other orientations, still the anisotropy remains. This result is changed by the rotational straining considered in §4.

From (3.18) and the probability distribution (3.12) for $(\alpha_1, \alpha_2, \alpha_3)$, we recover Batchelor's result (see Monin & Yaglom 1971, §24.5) for the ensemble average of the moments of vorticity over all orientations and magnitudes of the large-scale straining,

$$\langle\overline{\omega^2}(t)\rangle = \langle\overline{\omega_l\omega_l}(t)\rangle = \langle W_{ll}(t)\rangle = \exp[2\alpha_0^2 t^2]\langle\overline{\omega^2}(0)\rangle. \tag{3.23a}$$

Thus the mean-square vorticity increases exponentially in time, in the absence of viscous dissipation. The development of the ensemble average of the anisotropy tensor [which is defined as in (3.17) but without the angle suffixes] is given by

$$a_{ij}(t) = \{\tfrac{2}{5}+\tfrac{3}{5}\exp[-\tfrac{3}{2}\alpha_0^2 t^2]\}\,a_{ij}(0), \tag{3.23b}$$

which shows that the anisotropy of vorticity decreases exponentially to its limiting value, which is a fixed proportion, $\frac{2}{5}$, of its initial anisotropy (see figure 5). Note that from (3.23a, b) and (3.17), the vorticity moments can be obtained. An example is worked out in §5.

3.3. *Ensemble-averaged velocity moments*

The ensemble averages of the velocity moments over the orientation θ, ϕ, ψ of the straining motions can be obtained from (3.15). We define a velocity-moment tensor, which can be expressed in terms of $\Phi_{ij}(\boldsymbol{\chi}, t)$ as

$$\langle\overline{u_i u_j}(t)\rangle_{\theta\phi\psi} = \langle B_{ij}(t)\rangle_{\theta\phi\psi} = \left\langle \int \Phi_{ij}(\boldsymbol{\chi}, t)\,\mathrm{d}\boldsymbol{\chi}\right\rangle_{\theta\phi\psi}. \tag{3.24}$$

In order to calculate the evolution of the velocity field, it is convenient to define a new wavenumber velocity-moment tensor

$$\langle C_{ij}(t)\rangle_{\theta\phi\psi} = \left\langle \int \frac{\chi_i\chi_j}{\chi^2}\Phi_{ll}(\boldsymbol{\chi}, t)\,\mathrm{d}\boldsymbol{\chi}\right\rangle_{\theta\phi\psi}. \tag{3.25}$$

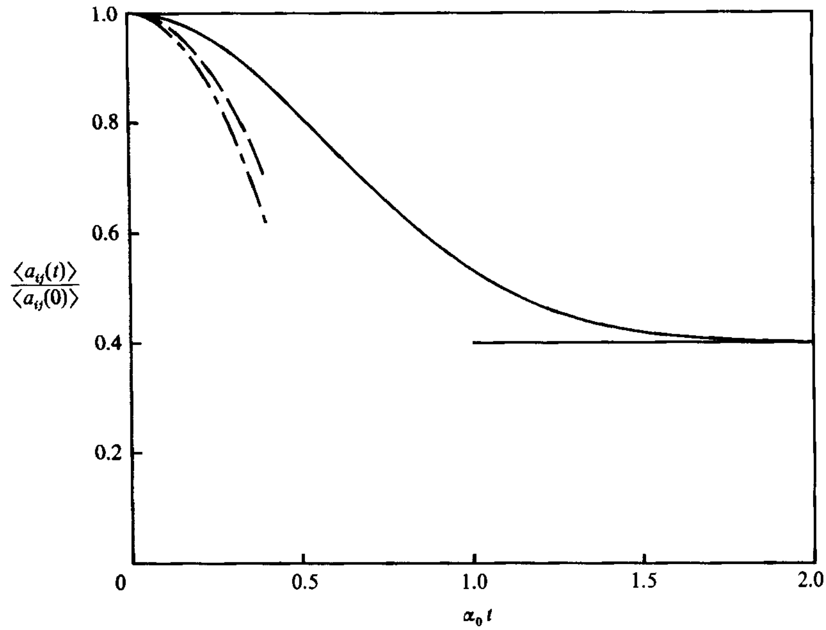

FIGURE 5. The small-scale anisotropic turbulence being distorted by large-scale isotropic turbulence, leading to the decay of anisotropy, defined by the anisotropy tensors for the vorticity, for different kinds of large-scale straining: ——, irrotational; — —, rotational ($\Omega_0^2 = \alpha_0^2$); – – –, rotational ($\Omega_0^2 = \frac{3}{2}\alpha_0^2$ as for isotropic large-scale straining).

For isotropic turbulence $\langle C_{ij}\rangle_{\theta\phi\psi} = \frac{1}{3}\delta_{ij}\langle\overline{u_l u_l}\rangle$. The main contribution to this tensor for any turbulent velocity field is from the eddies containing most energy (here we are interested in the small-scale turbulence). The principal axes of the tensor $\langle C_{ij}\rangle/\langle B_{ll}\rangle$ indicate which directions in wavenumber space contribute most and least at high wavenumber to the turbulent kinetic energy per unit mass $\frac{1}{2}q^2(t)$, which we define as

$$q^2(t) = \langle\overline{u_l u_l}\rangle = \langle B_{ll}(t)\rangle_{\theta\phi\psi} = \langle C_{ll}(t)\rangle_{\theta\phi\psi}. \tag{3.26}$$

For most kinds of spectra, the diagonal elements of $\langle C_{ij}\rangle/\langle B_{ll}\rangle$ can be estimated in terms of the integral scales of the turbulence in the i-directions; thus

$$\langle C_{ij}\rangle \approx \langle\overline{u_l u_l}\rangle \Big/ \left(L_i L_j \sum_{k=1}^{3} L_k^{-2}\right),$$

where
$$L_k = \int \phi_{ll}(\boldsymbol{\chi})\,\delta(\chi_k)\,\mathrm{d}\boldsymbol{\chi} \Big/ \int \phi_{ll}(\boldsymbol{\chi})\,\mathrm{d}\boldsymbol{\chi}.$$

In isotropic turbulence L_k is $\frac{2}{3}$ of L_x, as defined in the usual way,

$$L_x = \frac{1}{\overline{u_1^2}} \int_0^\infty \overline{u_1(x_1)\,u_1(x_1+r_1)}\,\mathrm{d}r_1.$$

To study the evolution of the anisotropy of the turbulence, we introduce the usual anisotropy velocity-moment tensor (Lumley 1978) defined in terms of $\langle B_{ij}\rangle$ as

$$b_{ij}(t) = \frac{\langle B_{ij}(t)\rangle_{\theta\phi\psi}}{\langle B_{ll}(t)\rangle_{\theta\phi\psi}} - \tfrac{1}{3}\delta_{ij}, \tag{3.27}$$

and the wavenumber anisotropy velocity tensor,

$$c_{ij}(t) = \frac{\langle C_{ij}(t)\rangle_{\theta\phi\psi}}{\langle C_{ll}(t)\rangle_{\theta\phi\psi}} - \tfrac{1}{3}\delta_{ij}. \tag{3.28}$$

For example, in a turbulent shear flow where the mean velocity

$$U = ((\mathrm{d}U_1/\mathrm{d}x_3)\,x_3, 0, 0), \quad \overline{u_3^2} < \overline{u_2^2} < \overline{u_1^2},$$

so $b_{33} < 0$ and $b_{11} > 0$. But the largest contribution in wavenumber space to the energy occurs where χ_3 is large and χ_1 is small, so $c_{33} > 0$, and $c_{11} < 0$ (Townsend 1976, p. 83).

By integrating (3.15) with respect to the wavenumber, we obtain the following expressions for q^2, b_{ij} and c_{ij} in terms of their initial values, without any other assumptions about the anisotropy of the initial small-scale energy-spectrum tensor $\Phi_{ij}(\boldsymbol{\kappa}, 0)$ (this is more general than previous rapid-distortion calculations, where it has been assumed that Φ_{ij} is isotropic or axisymmetric):

$$q^2(t) = F_q(t)\,q^2(0), \tag{3.29}$$

$$b_{ij}(t) = F_{11}(t)\,b_{ij}(0) + F_{12}(t)\,c_{ij}(0), \tag{3.30a}$$

$$c_{ij}(t) = F_{21}(t)\,b_{ij}(0) + F_{22}(t)\,c_{ij}(0). \tag{3.30b}$$

The general functions $F_q(t)$, $F_{11}(t)$, $F_{12}(t)$, $F_{21}(t)$ and $F_{22}(t)$ are functions of $\alpha_1 t$, $\alpha_2 t$, $\alpha_3 t$, and their explicit forms can be derived.†

When the strain is small, i.e. $\|\boldsymbol{\alpha}\|\,t \ll 1$, the expressions for F_q and $F_{11}, \ldots, F_{22}$ in (A 1) to (A 6) can be expanded in a Taylor series expansion. Then we take an ensemble average over all values of α_1, α_2, α_3. If we assume that the distribution of these strains is isotropic, then the full ensemble-average values of q^2, b_{ij}, c_{ij} can be calculated from (3.29) and (3.30), where now

$$\left.\begin{aligned} &F_q(t) = 1 + \tfrac{4}{5}\alpha_0^2 t^2 + \ldots, \\ &F_{11}(t) = 1 - \tfrac{9}{10}\alpha_0^2 t^2 + \ldots, \quad F_{12}(t) = o(\alpha_0^2 t^2), \\ &F_{22}(t) = 1 - \tfrac{9}{10}\alpha_0^2 t^2 + \ldots, \quad F_{21}(t) = o(\alpha_0^2 t^2), \end{aligned}\right\} \tag{3.31}$$

where $\alpha_0^2 = \langle \alpha_1^2 \rangle = \langle \alpha_2^2 \rangle = \langle \alpha_3^2 \rangle$. Note that for this small-time expansion, the form of the probability distribution of $\alpha_1, \alpha_2, \alpha_3$ is irrelevant, and also that the initial reduction of the anisotropy of the velocity moments $b_{ij}(t)$ is independent of $c_{ij}(0)$, and the initial reduction of the anisotropy of the wavenumber distribution $c_{ij}(t)$ is independent of $b_{ij}(0)$.

For large $\alpha_0 t$, using the expressions in Appendix A, it is found that

$$F_q \to \tfrac{3}{2}\exp[\tfrac{1}{2}\alpha_0^2 t^2], \quad F_{11} \to -\tfrac{1}{6}, \quad F_{12} \to \tfrac{5}{12}, \quad F_{21} \to \tfrac{1}{2}, \quad F_{22} \to -\tfrac{1}{4}.$$

Thence from (3.29) and (3.30)

$$q^2(t) \to \tfrac{3}{2}\exp[\tfrac{1}{2}\alpha_0^2 t^2]\,q^2(0), \tag{3.32}$$

$$b_{ij}(t) - c_{ij}(t) \to -\tfrac{2}{3}[b_{ij}(0) - c_{ij}(0)], \tag{3.33a}$$

$$6b_{ij}(t) + 5c_{ij}(t) \to \tfrac{1}{4}[6b_{ij}(0) + 5c_{ij}(0)]. \tag{3.33b}$$

Since $|-\tfrac{2}{3}| < 1$ and $|\tfrac{1}{4}| < 1$, this result can be interpreted as showing that the isotropic

† Details are provided in Appendix A. This and the other appendices mentioned in the text are held in the JFM files, and copies may be obtained from the authors or the Editor on request.

straining leads to a reduction in an overall measure of anisotropy which combines the anisotropy in the direction of the velocity components and the anisotropy in the directions in wavenumber space contributing to the velocity. As in the anisotropy of vorticity in (3.23*b*), there is a lower limit to this reduction.

If we want to separate these types of anisotropy we find that when $\alpha_0 t \gg 1$, $b_{ij}(t) \rightarrow -\frac{1}{6}b_{ij}(0)+\frac{5}{12}c_{ij}(0)$ and $c_{ij}(t) \rightarrow \frac{1}{2}b_{ij}(0)-\frac{1}{4}c_{ij}(0)$. This result should not really be surprising since the anisotropy of the velocity components depends on the initial anisotropy of the velocity components and on the anisotropy of the wavenumber distributions. This also supports the recent results of Lee & Reynolds (1985) that the development of anisotropy cannot be defined wholly in terms of the anisotropy of the velocity moments, as is assumed (for simplicity) in many turbulence models (e.g. Lumley 1978).

4. Isotropic rotational large-scale turbulence

4.1. *Vorticity analysis*

Now we consider the effect of the large-scale strain having a significant rotational part $\boldsymbol{\alpha}^{(\mathrm{R})}$, when averaged over a scale L^*. As we mentioned in the Introduction, $\boldsymbol{\alpha}^{(\mathrm{R})}$ is likely to be induced by the combined effect of many smaller-scale regions of intense vorticity.

The rate-of-strain tensor can always be divided into its irrotational and rotational (or symmetric and antisymmetric) components as:

$$\boldsymbol{\alpha} = \boldsymbol{\alpha}^{(\mathrm{I})}+\boldsymbol{\alpha}^{(\mathrm{R})}, \tag{4.1}$$

where
$$(\boldsymbol{\alpha}^{(\mathrm{I})})_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right), \quad (\boldsymbol{\alpha}^{(\mathrm{R})})_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j}-\frac{\partial U_j}{\partial x_i}\right).$$

Let $\Omega_i = \epsilon_{ijk}\,\partial U_k/\partial x_j$ and $\Omega = (\Omega_i\,\Omega_i)^{\frac{1}{2}}$, then $\boldsymbol{\alpha}^{(\mathrm{R})}$ can be expressed in terms of the magnitude of the vorticity, Ω, and a rotation matrix $\boldsymbol{R}$, as

$$\boldsymbol{\alpha}^{(\mathrm{R})} = \tilde{\boldsymbol{R}}\cdot\begin{pmatrix} 0 & -\frac{1}{2}\Omega & 0 \\ \frac{1}{2}\Omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\cdot\boldsymbol{R}, \tag{4.2}$$

where $\boldsymbol{R}$ is given in terms of the angles Θ and Φ between $\boldsymbol{\Omega}$ and the (xyz)-axes (figure 6)

$$\boldsymbol{R} = \begin{pmatrix} \cos\Phi & \sin\Phi & 0 \\ -\cos\Theta\sin\Phi & \cos\Theta\cos\Phi & \sin\Theta \\ \sin\Theta\sin\Phi & -\sin\Theta\cos\Phi & \cos\Theta \end{pmatrix}. \tag{4.3}$$

As a first approximation we assume that the angles (θ,ϕ,ψ) and (Θ,Φ), which define the orientations of the irrotational and rotational large-scale motions, are distributed independently of each other. (This is certainly not true at the smallest scales of a fully turbulent flow; it may be a better approximation at larger scales.)

As is shown in Appendix B, for a homogeneous (pseudo-) isotropic large-scale turbulence the ensemble averages over all orientations and magnitudes of the first- and second-order moments of the rate-of-strain tensor $\boldsymbol{\alpha}$ are respectively given by

$$\langle\boldsymbol{\alpha}\rangle = 0 \tag{4.4a}$$

and
$$\langle\alpha_{ij}\,\alpha_{kl}\rangle = -\tfrac{1}{5}\alpha_0^2\,\delta_{ij}\,\delta_{kl}+(\tfrac{3}{10}\alpha_0^2+\tfrac{1}{3}\Omega_0^2)\,\delta_{ik}\,\delta_{jl}+(\tfrac{3}{10}\alpha_0^2-\tfrac{1}{3}\Omega_0^2)\,\delta_{il}\,\delta_{jk}, \tag{4.4b}$$

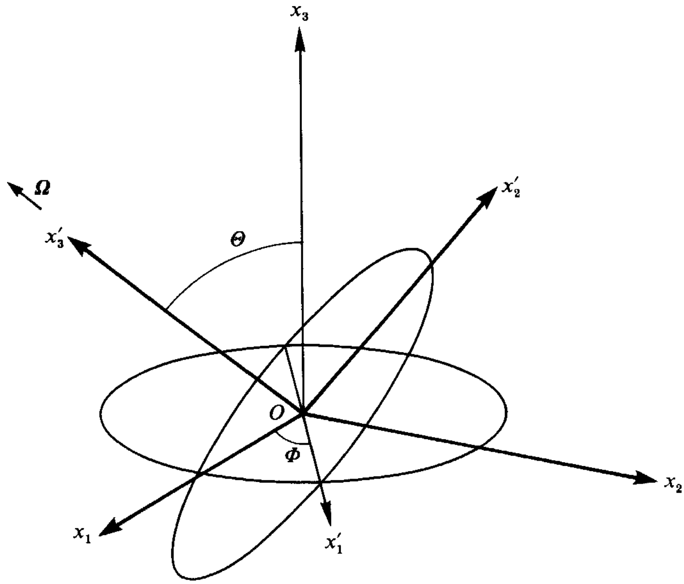

FIGURE 6. The coordinate systems used for analysing different orientations of rotational straining. $\boldsymbol{\Omega}$ is the vector of the vorticity of the large-scale straining.

where $\Omega_0^2 \equiv \frac{1}{4}\langle \Omega^2 \rangle$ is a quarter of the mean-square vorticity of the large-scale motion. Note that if the large-scale turbulence is isotropic in the strict sense, we have $\Omega_0^2 = \frac{3}{2}\alpha_0^2$ and

$$\langle \alpha_{ij}\alpha_{kl}\rangle = -\tfrac{1}{5}\alpha_0^2(\delta_{ij}\delta_{kl} - 4\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{4.4c}$$

The ensemble average of the vorticity-moment tensor (3.16) can be calculated for small strains as follows. Substitution of (2.25) into (3.16) yields

$$\langle \overline{\omega_i \omega_j}(t)\rangle = \langle W_{ij}(t)\rangle = \int \langle T_{ia}(t)\,T_{jb}(t)\rangle\,\Omega_{ab}(\kappa, 0)\,\mathrm{d}\kappa. \tag{4.5}$$

For small strains the vorticity deformation matrix $\boldsymbol{T}$, which is defined by (2.19a), is expanded in terms of t as

$$T_{ij}(t) = \delta_{ij} + \left[\alpha_{ji} + \frac{\kappa_i\kappa_a}{\kappa^2}(\alpha_{aj} - \alpha_{ja})\right]t + \frac{1}{2}\left\{\left(\frac{2\kappa_i\kappa_a\kappa_b\kappa_c\alpha_{ac}}{\kappa^4} - \frac{\kappa_i\kappa_a\alpha_{ab}}{\kappa^2} - \frac{\kappa_a\kappa_b\alpha_{ai}}{\kappa^2}\right)\alpha_{bj} - \alpha_{jb}) + \left[\alpha_{ci} + \frac{\kappa_i\kappa_a}{\kappa^2}(\alpha_{ac} - \alpha_{ca})\right]\left[\alpha_{jc} + \frac{\kappa_b\kappa_c}{\kappa^2}(\alpha_{bj} - \alpha_{jb})\right]\right\}t^2 + \dots. \tag{4.6}$$

By substituting (4.6) into (4.5) and taking an ensemble average over $\boldsymbol{\alpha}$, we obtain

$$\langle \overline{\omega_i \omega_j}(t)\rangle = \langle W_{ij}(t)\rangle = \{1 + (\tfrac{11}{10}\alpha_0^2 - \Omega_0^2)\,t^2\}\langle W_{ij}(0)\rangle + (\tfrac{3}{10}\alpha_0^2 + \tfrac{1}{3}\Omega_0^2)\,t^2\,\delta_{ij}\langle W_{ll}(0)\rangle + \dots \tag{4.7a}$$

and $$\langle \overline{\omega_l \omega_l}(t)\rangle = \langle W_{ll}(t)\rangle = (1 + 2\alpha_0^2 t^2 + \dots)\langle W_{ll}(0)\rangle, \tag{4.7b}$$

where use has been made of (4.4a, b).

The vorticity anisotropy tensor (3.17) is then written as

$$a_{ij}(t) = \{1 - (\tfrac{9}{10}\alpha_0^2 + \Omega_0^2)\,t^2 + \dots\}\,a_{ij}(0). \tag{4.7c}$$

To extend these calculations to strains of $O(1)$ requires computation of the relevant integrals.

Similarly the ensemble averages of the velocity-moment tensor and anisotropy velocity tensor are calculated to be

$$\langle \overline{u_i u_j}(t)\rangle = B_{ij}(t) = \{1-(\tfrac{1}{10}\alpha_0^2+\Omega_0^2)\,t^2\}B_{ij}(0)+(\tfrac{3}{10}\alpha_0^2+\tfrac{1}{3}\Omega_0^2)\,t^2\,\delta_{ij}B_{ll}(0)+\dots, \quad (4.8a)$$

$$q^2(t) = B_{ll}(t) = \{1+\tfrac{4}{5}\alpha_0^2 t^2+\dots\}B_{ll}(0) \quad (4.8b)$$

and

$$b_{ij}(t) = \{1-(\tfrac{9}{10}\alpha_0^2+\Omega_0^2)\,t^2+\dots\}\,b_{ij}(0). \quad (4.9)$$

These results show that for strains of $O(\alpha_0^2 t^2)$ or $O(\Omega_0^2 t^2)$, the rotational part of the straining motion does not contribute to the amplification of the mean-square vorticity (enstrophy) or the mean-square velocity, but that it does help to reduce the anisotropy by the same order as the irrotational straining motion (see figure 5 and the next section for an example). The explanation for (4.7)–(4.9) is that, since the rotational part gives a rigid rotation as the leading-order effect, the lengths of the vortex lines and magnitudes of the wavenumbers are not distorted. However this rotation tends to reduce the anisotropy in $\boldsymbol{\omega}$ and $\boldsymbol{u}$.

The analysis has not yet been extended to the case where $\alpha_0^2 t^2$ or $\Omega_0^2 t^2$ are $O(1)$, but we expect that, as in the simple example discussed in §3.3, the effect of rotation would ensure that isotropic straining could lead to approximately isotropic small-scale turbulence. Note that at larger times the effect of the initial distribution of energy in wavenumber space becomes more important. Further research to examine this conjecture is necessary.

In estimating the energy transfer between turbulence with scales larger than k^{-1} and scales smaller than k^{-1}, Townsend (1976, p. 99) only considered the symmetric part of $\boldsymbol{\alpha}$. These results give some support to that assumption.

4.2. *Calculations using pressure and the momentum equation*

An alternative method for calculating the change in the energy of small-scale turbulence under the action of large-scale strainings is to use the approach outlined in §2.4. The expressions (2.35b) and (2.37) do not give $\langle P_{\mathrm{RS}}\rangle$ and $\langle P_{\mathrm{PG}}\rangle$ to $O(t^2)$ because u_i is a function of $\partial U_i/\partial x_k$. By expanding $\Phi_{ij}(\boldsymbol{\chi},t)$ and $\boldsymbol{\chi}$ in terms of t, taking an ensemble average of (2.35b) and (2.37) over all orientations and magnitudes of $\boldsymbol{\alpha}$ and integrating with respect to the wavenumber, we obtain

$$\langle P_{\mathrm{RS}}\rangle_{ij} = t\{(\alpha_0^2-\tfrac{2}{3}\Omega_0^2)B_{ij}(0)+(\tfrac{3}{5}\alpha_0^2+\tfrac{2}{3}\Omega_0^2)(\delta_{ij}B_{ll}(0)-2C_{ij}(0))\}, \quad (4.10a)$$

and

$$\langle P_{\mathrm{PG}}\rangle_{ij} = -t(\tfrac{6}{5}\alpha_0^2+\tfrac{4}{3}\Omega_0^2)(B_{ij}(0)-C_{ij}(0)). \quad (4.10b)$$

Now $\overline{u_i u_j}$ can be computed to $O(t^2)$ by integrating these expressions (see (2.35a)). These results are consistent with (4.8a).

We see that both the Reynolds-stress production terms $\langle P_{\mathrm{RS}}\rangle$ and the pressure terms $\langle P_{\mathrm{PG}}\rangle$ are functions of the distribution of energy in wavenumber space as defined by the complementary tensor C_{ij}. This effect is omitted in all current models of turbulence for the second moments of the velocity.

Consider the model of Launder *et al.* (1975) applied to this rather pathological problem (which is certainly outside the range of flows for which the model was derived). In that model, in the absence of a mean velocity gradient,

$$\langle P_{\mathrm{PG}}\rangle_{ij} \approx c_1\frac{\epsilon}{\overline{U_l^* U_l^*}}\left(\tfrac{1}{3}\delta_{ij}-\frac{\overline{U_i^* U_j^*}}{\overline{U_l^* U_l^*}}\right), \quad (4.11)$$

where ϵ is the rate of dissipation of turbulent energy per unit mass, U_i^* is the total turbulent velocity, and c_1 is a constant. For this problem of energetic large-scale isotropic turbulence and small-scale anisotropic turbulence, $\epsilon \sim U'^3/L$, and $\overline{U_l^* U_l^*} \sim 3U'^2$. Therefore, since all the anisotropy is at small scale, (4.11) implies

$$\langle P_{\mathrm{PG}} \rangle_{ij} \sim c_1 \frac{U'}{L}\left(\tfrac{1}{3}\delta_{ij} - \frac{\overline{u_i u_j}}{q^2}\right) = c_1 \frac{U'}{L} b_{ij}. \tag{4.12}$$

Substituting this expression into (2.35 *a*) we can see that this model implies that the rate of return to isotropy is given by

$$b_{ij}(t) \approx b_{ij}(0)\left(1 - t\frac{c_1 U'}{L}\right). \tag{4.13}$$

Contrast this result for (4.13) with the exact result (for small time) in (4.9), which shows that the change in b_{ij} is of order $(U'^2/L^2)\,t^2 b_{ij}(0)$. So the change predicted by this theory is initially slower than the estimate derived from the usual second-order models, or from spectral models which have a relaxation process, such as EDQNM (Bertoglio 1986).

Compare also the important difference between the exact expression for P_{PG} at small times and the modelled term in (4.12). Not only is there a different dependence on time, but there is a difference in the dependence on the distribution of energy in wavenumber space of the small-scale turbulence. In (4.10 *b*) there is such a dependence, but there is not in (4.12). At first sight this looks like an important difference, but in fact it may not be so important because in all models the effects of C_{ij} on the Reynolds-stress terms are also overlooked. As we have seen, to $O(t)$, the contributions by C_{ij} in these two terms exactly cancel! That is why $b_{ij}(t)$ is independent of c_{ij} to $O(t^2)$.

5. Examples of large- and small-scale interactions

5.1. *Two-dimensional isotropic small-scale turbulence disturbed by isotropic large-scale turbulence*

Consider a situation where the small-scale turbulence is initially two-dimensional but isotropic in the $(Ox_1 x_2)$-plane, so that $\overline{\omega_1^2} = \overline{\omega_2^2} = 0$ and $\overline{\omega_3^2} \neq 0$, while $\overline{u_1^2} = \overline{u_2^2} (= \frac{1}{2}\overline{u_l u_l})$ and $\overline{u_3^2} = 0$. Then the initial vorticity and velocity anisotropy tensors have diagonal components:

vorticity: $$a_{11} = a_{22} = -\tfrac{1}{3}, \qquad a_{33} = 1 - \tfrac{1}{3} = \tfrac{2}{3}, \tag{5.1a}$$

velocity: $$b_{11} = b_{22} = \tfrac{1}{2} - \tfrac{1}{3} = \tfrac{1}{6}, \qquad b_{33} = -\tfrac{1}{3}. \tag{5.1b}$$

The components of the complementary velocity-moment tensor C_{ij} defined by (3.25) for two-dimensional turbulence are such that $C_{11} = C_{22}$, and $C_{33} = 0$. So the initial wavenumber anisotropy tensor c_{ij} has diagonal components:

$$c_{11} = c_{22} = \tfrac{1}{6}, \qquad c_{33} = -\tfrac{1}{3}. \tag{5.1c}$$

We first consider the effects of random irrotational straining. From the solution (3.23 *b*) it follows that the vorticity anisotropy tensor decreases exponentially to $\frac{2}{5}$ of its initial value:

$$a_{11} = a_{22} \rightarrow \tfrac{2}{5} \times (-\tfrac{1}{3}) = -\tfrac{2}{15}, \qquad a_{33} \rightarrow \tfrac{2}{5} \times \tfrac{2}{3} = \tfrac{4}{15}. \tag{5.2}$$

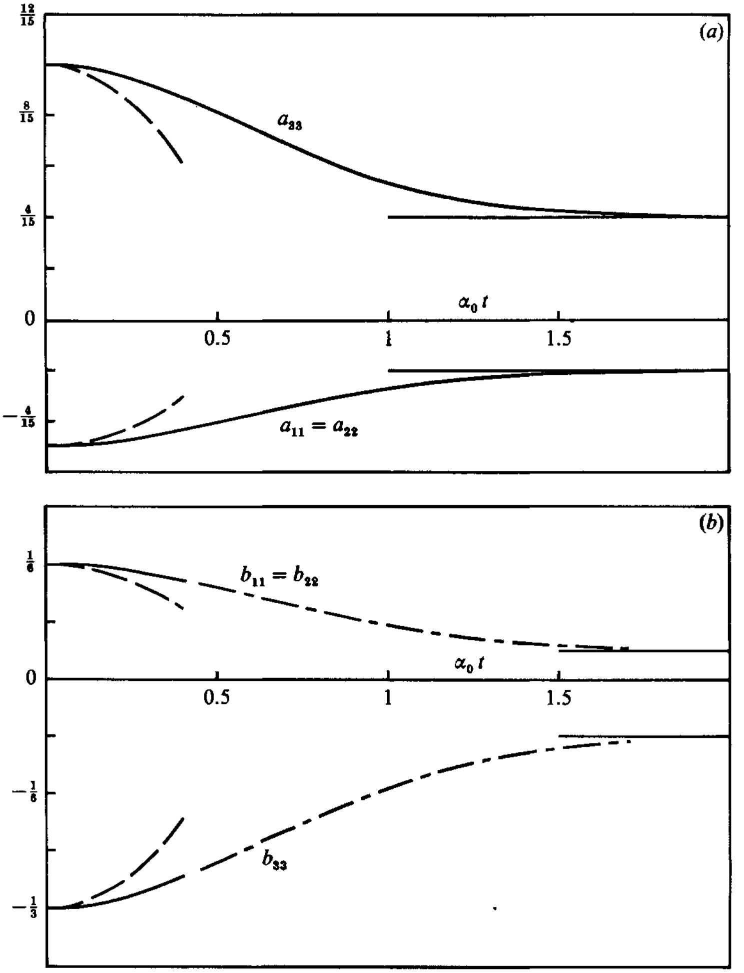

FIGURE 7. An example of small-scale anisotropic turbulence being distorted by large-scale isotropic turbulence, to show the decrease in anisotropy for a particular example where initially $\overline{u_1^2} = \overline{u_2^2}, \overline{u_3^2} = 0$. (*a*) The vorticity anisotropy tensor a_{ij}. (*b*) The anisotropy velocity-moment tensor b_{ij}. (Only the initial and final forms are given for irrotational strains; only the initial form for rotational strains.) ——, irrotational; — —, rotational ($\Omega_0^2 = \frac{3}{2}\alpha_0^2$). Note that the wavenumber anisotropy velocity-moment tensor c_{ij} has the same form as b_{ij} (at least with irrotational straining).

Thence from (3.17) and (3.23*a*) the asymptotic form of $\overline{\omega_i \omega_j}$ is

$$\left.\begin{aligned} \langle\overline{\omega_1^2}\rangle(t) = \langle\overline{\omega_2^2}\rangle(t) &\to \tfrac{1}{5}\exp[2\alpha_0^2 t^2]\langle\overline{\omega_1^2}\rangle(0), \\ \langle\overline{\omega_1^2}\rangle(t) &\to \tfrac{3}{5}\exp[2\alpha_0^2 t^2]\langle\overline{\omega_1^2}\rangle(0). \end{aligned}\right\} \quad (5.3)$$

From the solution (4.8*a*), the initial changes in the components of the velocity-moment tensor are

$$\overline{u_1^2}(t) = \overline{u_2^2}(t) = (1 + \tfrac{1}{2}\alpha_0^2 t^2)\,\overline{u_1^2}(0), \qquad \overline{u_3^2}(t) = \tfrac{3}{5}\alpha_0^2 t^2 \overline{u_1^2}(0). \quad (5.4)$$

Asymptotically, as $\alpha_0 t \gg 1$, $\overline{u_1^2}$ and $\overline{u_2^2}$ increase exponentially and their anisotropy tensors take the limiting form (from (3.32) and (3.33)):

$$b_{11} = b_{22} \to -\tfrac{1}{6}\tfrac{1}{6} + \tfrac{5}{12}\tfrac{1}{6} = \tfrac{1}{24}, \qquad b_{33} \to -\tfrac{1}{6}(-\tfrac{1}{3}) + \tfrac{5}{12}(-\tfrac{1}{3}) = -\tfrac{1}{12}, \quad (5.5a)$$

and

$$c_{11} = c_{22} \to \tfrac{1}{2}\tfrac{1}{6} - \tfrac{1}{4}\tfrac{1}{6} = \tfrac{1}{24}, \qquad c_{33} \to \tfrac{1}{2}(-\tfrac{1}{3}) - \tfrac{1}{4}(-\tfrac{1}{3}) = -\tfrac{1}{12}. \quad (5.5b)$$

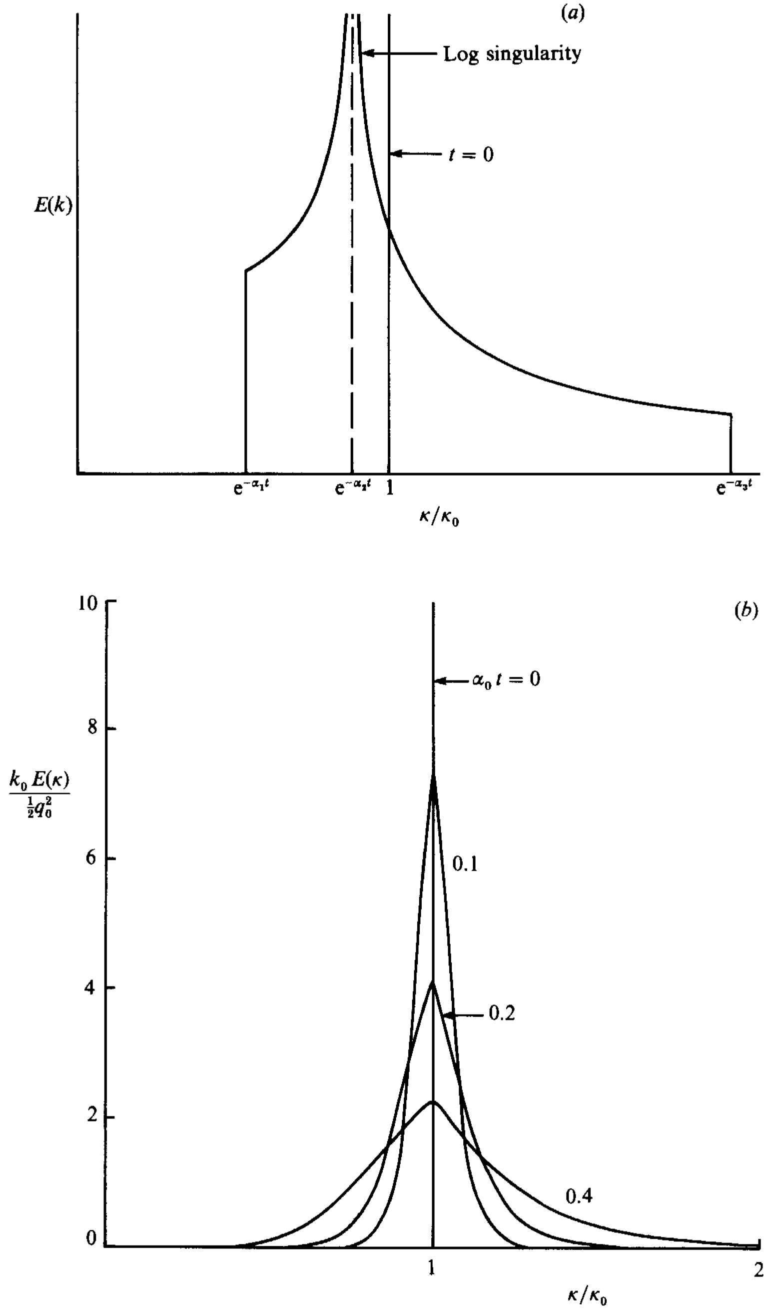

FIGURE 8. The change in a delta-function spectrum of small-scale turbulence undergoing random irrotational straining by large-scale turbulence. (Rotational straining does not affect $E(\kappa)$ over short times.) (a) The fixed magnitudes of principal strains, but random orientations, $\alpha_1 > \alpha_2 > \alpha_3$, where $\alpha_2 > 0$; (b) Gaussian distribution of strains with variance α_0^2.

Thus the effect of isotropic irrotational straining increases the energy and vorticity of the small-scale turbulence, and reduces its anisotropy but cannot completely eliminate it. Note that the asymptotic forms for the components are approximately valid even when $\alpha_0 t \gtrsim 2$.

We now consider the effect of random rotational strainings using the results of §4. The solution (4.7*c*) gives the change in the anisotropy tensor:

$$a_{ij}(t) = [1-(\tfrac{9}{10}\alpha_0^2+\Omega_0^2)\,t^2]\,a_{ij}(0), \tag{5.6}$$

showing that the rate of reduction of anisotropy of vorticity is faster.

Likewise the rate of change in the anisotropy of the velocity tensor is increased:

$$\left.\begin{aligned}\overline{u_1^2}(t) = \overline{u_2^2}(t) &= [1+(\tfrac{1}{2}\alpha_0^2-\tfrac{1}{3}\Omega_0^2)\,t^2]\,\overline{u_1^2}(0),\\ \overline{u_3^2}(t) &= (\tfrac{3}{5}\alpha_0^2+\tfrac{2}{3}\Omega_0^2)\,t_2\overline{u_1^2}(0).\end{aligned}\right\} \tag{5.7}$$

These differences are shown on figure 7. Note here that the change in the mean-square vorticity $\overline{\omega_l\,\omega_l}$ and energy q^2 is not affected by the rotational straining.

5.2. *The change in the spectrum*

When small-scale turbulence is distorted by large-scale turbulence, its spectrum is also changed as well as its energy and its anisotropy. This aspect of the interaction has been of special interest in the experimental studies e.g. by Kellogg & Corrsin (1980) and others.

We consider the idealized problem of isotropic small-scale turbulence whose initial spectrum is a delta function being distorted by random irrotational isotropic large-scale straining. The mathematical problem is summarized by (3.15). Since the initial spectrum is isotropic and the straining is isotropic, the spectrum remains isotropic, so

$$\Phi_{ij}(\chi,t) = \frac{E(\chi,t)}{4\pi\chi^4}(\chi^2\delta_{ij}-\chi_i\chi_j), \tag{5.8a}$$

where, at $t=0$,

$$E(\kappa,0) = \tfrac{1}{2}q_0^2\,\delta(\kappa-\kappa_0). \tag{5.8b}$$

The details of the calculation are omitted. In the first stage we consider how $E(\chi,t)$ changes under the action of random orientation of straining by fixed amplitude of straining ($\alpha_1 > \alpha_2 > \alpha_3$ and $\alpha_1 > 0, \alpha_3 < 0$); (figure 8*a*). The result is that $E(\chi,t)$ continues to be singular at the wavenumber $\chi = \kappa_0\exp[-\alpha_2 t]$. But the spectrum spreads towards both low and high wavenumbers. At a given value of time the spectrum lies between the limits $\kappa_0\exp[-\alpha_1 t]$ and $\kappa_0\exp[-\alpha_3 t]$. The exact form of $E(\chi,t)$ is given by elliptic functions:

$$E(\chi,t) = \begin{cases} \dfrac{\frac{1}{2}q_0^2}{\pi\kappa_0}\left(-\left\{[\mathrm{e}^{2\alpha_2 t}-\mathrm{e}^{2\alpha_3 t}]\left[\mathrm{e}^{2\alpha_1 t}-\left(\dfrac{\kappa_0}{\chi}\right)^2\right]\right\}^{\frac{1}{2}} Y(r) \right. \\ \qquad \left. +\dfrac{\left(\dfrac{\kappa_0}{\chi}\right)^2\mathrm{e}^{2\alpha_3 t}+\mathrm{e}^{-2\alpha_3 t}}{\left\{[\mathrm{e}^{2\alpha_2 t}-\mathrm{e}^{2\alpha_3 t}]\left[\mathrm{e}^{2\alpha_1 t}-\left(\dfrac{\kappa_0}{\chi}\right)^2\right]\right\}^{\frac{1}{2}}}X(r)\right) & \text{for } \kappa_0\mathrm{e}^{-\alpha_2 t} < \chi \leqslant \kappa_0\mathrm{e}^{-\alpha_3 t}, \\ \dfrac{\frac{1}{2}q_0^2}{\pi\kappa_0}\left(-\left\{[\mathrm{e}^{2\alpha_1 t}-\mathrm{e}^{2\alpha_2 t}]\left[\left(\dfrac{\kappa_0}{\chi}\right)^2-\mathrm{e}^{2\alpha_3 t}\right]\right\}^{\frac{1}{2}} Y\left(\dfrac{1}{r}\right)\right. \\ \qquad \left. +\dfrac{\left(\dfrac{\kappa_0}{\chi}\right)^2\mathrm{e}^{2\alpha_1 t}+\mathrm{e}^{-2\alpha_1 t}}{\left\{[\mathrm{e}^{2\alpha_1 t}-\mathrm{e}^{2\alpha_2 t}]\left[\left(\dfrac{\kappa_0}{\chi}\right)^2-\mathrm{e}^{2\alpha_3 t}\right]\right\}^{\frac{1}{2}}}X\left(\dfrac{1}{r}\right)\right) & \text{for } \kappa_0\mathrm{e}^{-\alpha_1 t} \leqslant \chi < \kappa_0\mathrm{e}^{-\alpha_2 t}, \\ 0 \quad \text{otherwise,} \end{cases} \tag{5.9a}$$

where $X(r)$ and $Y(r)$ are the complete elliptic functions of the first and second kinds, respectively:

$$X(r) = \int_0^1 \frac{\mathrm{d}s}{[(1-s^2)(1-r^2s^2)]^{\frac{1}{2}}} \quad Y(r) = \int_0^1 \frac{(1-r^2s^2)^{\frac{1}{2}}}{1-s^2}\mathrm{d}s \tag{5.9b}$$

and

$$r = \frac{(\mathrm{e}^{2\alpha_1 t}-\mathrm{e}^{2\alpha_2 t})\left(\left(\frac{\kappa_0}{\chi}\right)^2-\mathrm{e}^{2\alpha_3 t}\right)^{\frac{1}{2}}}{\left(\mathrm{e}^{2\alpha_1 t}-\left(\frac{\kappa_0}{\chi}\right)^2\right)(\mathrm{e}^{2\alpha_2 t}-\mathrm{e}^{2\alpha_3 t})}. \tag{5.9c}$$

Here we have assumed $\alpha_1 > \alpha_2 > \alpha_3$ without loss of generality. Note that if α_2 is positive (a straining by 'squashing'), then the maximum moves to low wavenumbers because more fluid elements are lengthened than compressed. If α_2 is negative (or a straining by 'elongation') the maximum moves to high wavenumbers because more fluid elements are compressed.

When we consider a Gaussian distribution of straining motion, the peak in the spectrum is finite and decreases with time. In fact for small initial strains $\alpha_0 t \ll 1$, the maximum of $E(\kappa_0, t)$ decreases rapidly as

$$E(\kappa_0, t) \propto \frac{q_0^2}{\kappa_0 \alpha_0 t}, \tag{5.10}$$

and the width of the peak increases in proportion to $\kappa_0 \alpha_0 t$. A computed spectrum is shown in figure 8(*b*). At larger times the centre of gravity of the spectrum moves towards high wavenumbers since the high wavenumbers are preferentially amplified by the random straining motions.

5.3. *Interaction between isotropic small-scale turbulence and non-isotropic large-scale turbulence*

We now consider small-scale isotropic turbulence undergoing straining by a strain field $\boldsymbol{\alpha}$, which is anisotropic. It is possible to calculate the initial development of the velocity-moment tensor $B_{ij}(t)$ (to $O(\alpha_0^2 t^2)$) by integrating (2.27) with respect to χ after the vorticity deformation tensor $\boldsymbol{T}(t)$ and the wavenumber $\chi(t)$ are expanded in terms of small time t. (The details are given in Appendices C and D.)

The general result for the velocity-moment tensor when the large-scale straining is anisotropic and rotational is

$$B_{ij}(t) = \frac{2}{3}\int E(k,0)\,\mathrm{d}k\left\{\delta_{ij}+\frac{t^2}{35}[8\langle\boldsymbol{\alpha}^2\rangle_{aa}\delta_{ij}+8\langle\alpha_{ab}\alpha_{ab}\rangle\delta_{ij}-5\langle\boldsymbol{\alpha}^2\rangle_{ij} -5\langle\boldsymbol{\alpha}^2\rangle_{ji}-12\langle\alpha_{ai}\alpha_{aj}\rangle+2\langle\alpha_{ia}\alpha_{ja}\rangle]\right\}. \tag{5.11a}$$

The energy is

$$B_{ll}(t) = 2\int E(k,0)\,\mathrm{d}k[1+\tfrac{2}{15}t^2(\langle\boldsymbol{\alpha}^2\rangle_{aa}+\langle\alpha_{ab}\alpha_{ab}\rangle)], \tag{5.11b}$$

and the anisotropy tensor is

$$b_{ij}(t) = \frac{t^2}{315}[10(\langle\boldsymbol{\alpha}^2\rangle_{aa}+\langle\alpha_{ab}\alpha_{ab}\rangle)\delta_{ij}-3(5\langle\boldsymbol{\alpha}^2\rangle_{ij}+5\langle\boldsymbol{\alpha}^2\rangle_{ji}+12\langle\alpha_{ai}\alpha_{aj}\rangle-2\langle\alpha_{ia}\alpha_{ja}\rangle)]. \tag{5.11c}$$

As an example, consider straining motions that are axisymmetric about the axis $\lambda_i = \delta_{i3}$. This is of importance near interfaces where small-scale turbulence is distorted by large-scale eddies impinging on the interface (Hunt 1984). In this case we have

$$B_{ij}(t) = \frac{2}{3}\int E(k,0)\,\mathrm{d}k\Big\{\delta_{ij}+\frac{t^2}{35}[(24\langle\alpha_{11}^2\rangle+13\langle\alpha_{33}^2\rangle+18\langle\alpha_{13}^2\rangle$$
$$+4\langle\alpha_{31}^2\rangle+22\langle\alpha_{13}\alpha_{31}\rangle)\,\delta_{ij}+(40\langle\alpha_{11}^2\rangle-25\langle\alpha_{33}^2\rangle-26\langle\alpha_{13}^2\rangle$$
$$+16\langle\alpha_{31}^2\rangle-10\langle\alpha_{13}\alpha_{31}\rangle)\,\lambda_i\lambda_j]\Big\}, \tag{5.12a}$$

$$B_{ii}(t) = 2\int E(k,0)\,\mathrm{d}k[1+\tfrac{2}{15}t^2(8\langle\alpha_{11}^2\rangle+\langle\alpha_{33}^2\rangle+2\langle\alpha_{13}^2\rangle+2\langle\alpha_{31}^2\rangle+4\langle\alpha_{13}\alpha_{31}\rangle)], \tag{5.12b}$$

$$b_{ij}(t) = \frac{t^2}{315}(-40\langle\alpha_{11}^2\rangle+25\langle\alpha_{33}^2\rangle+26\langle\alpha_{13}^2\rangle-16\langle\alpha_{31}^2\rangle+10\langle\alpha_{13}\alpha_{31}\rangle)\,(\delta_{ij}-3\lambda_i\lambda_j). \tag{5.12c}$$

Consider an interface of $z=0$, where $U_3 \rightarrow 0$. Since the characteristic length in the x_3 direction is smaller than those in the x_1 and x_2 directions near such an interface, we have the relation

$$\left|\frac{\partial U_1}{\partial x_3}\right| \gg \left|\frac{\partial U_1}{\partial x_1}\right| \approx \left|\frac{\partial U_2}{\partial x_2}\right| \approx \left|\frac{\partial U_3}{\partial x_3}\right| \gg \left|\frac{\partial U_3}{\partial x_1}\right|$$

or

$$\langle\alpha_{13}^2\rangle \gg \langle\alpha_{11}^2\rangle \approx \langle\alpha_{33}^2\rangle \gg \langle\alpha_{31}^2\rangle. \tag{5.13}$$

Therefore (5.12) leads to

$$\frac{\overline{u_1^2}(t)}{q^2(t)} = \tfrac{1}{3}(1+\tfrac{26}{105}\langle\alpha_{13}^2\rangle\,t^2), \quad q^2(t) = q^2(0)\,(1+\tfrac{4}{15}\langle\alpha_{13}^2\rangle\,t^2), \tag{5.14a}$$

while

$$\frac{\overline{u_3^2}(t)}{q^2(t)} = \tfrac{1}{3}(1-\tfrac{52}{105}\langle\alpha_{13}^2\rangle\,t^2). \tag{5.14b}$$

In other words the components of small-scale turbulence parallel to the surface are preferentially amplified compared to the normal component as observed in the experiments of Thomas & Hancock (1977) and the numerical simulations of Biringen & Reynolds (1981). See also the discussion by Hunt (1984).

6. Interaction of large-scale and small-scale turbulence in the presence of a mean strain

We now consider the changes in the vorticity and velocity of small-scale turbulence when it is being distorted by large-scale irrotational mean straining motions and by large-scale random straining motions. We want to explore how these two kinds of straining motions interact.

The same theory can be used as defined in §2.2, the only change being that now the large-scale straining motion $\Delta\boldsymbol{U}$ has a mean component $\boldsymbol{A}\cdot\boldsymbol{x}$ and a fluctuating component $\boldsymbol{\alpha}\cdot\boldsymbol{x}$, so $\Delta\boldsymbol{U} = \boldsymbol{A}^*\cdot\boldsymbol{x}$, where

$$\boldsymbol{A}^* = \boldsymbol{A}+\boldsymbol{\alpha}. \tag{6.1a}$$

We choose a coordinate system in which $\boldsymbol{A}$ is diagonalized, so that

$$A_{ij} = A^{(i)}\delta_{ij}. \tag{6.1b}$$

How does this correspond to an experimental flow? If a two-scale turbulent flow is initially generated and then distorted by a mean straining motion, the vorticity of the large-scale component is distorted as well as that of the small-scale. So the rotational component of $\boldsymbol{\alpha}$ is a function of $\boldsymbol{A}$ and changes with time. In fact $\boldsymbol{\alpha}$ can only be a stationary and isotropic random straining motion if it is continuously generated by some internal force field. So the first calculation where $\boldsymbol{\alpha}$ and $\boldsymbol{A}$ coexist is rather idealistic. In the second calculation we consider the small-scale turbulence to be first distorted by the mean motion and then by the large-scale fluctuation, e.g. by a small grid followed by a contraction followed by a large grid and a uniform flow.

The deformation matrix $\boldsymbol{S}$ and the vorticity deformation matrix $\boldsymbol{T}$ defined by (2.18a) and (2.19a) are expanded as power of time t, the initial wavenumber $\boldsymbol{\kappa}$ and the strain tensor $\boldsymbol{A}^*$. It is found that

$$\boldsymbol{T} = \boldsymbol{I} + \boldsymbol{T}^{(1)}t + \frac{1}{2!}\boldsymbol{T}^{(2)}t^2 + \frac{1}{3!}\boldsymbol{T}^{(3)}t^3 + \ldots, \tag{6.2a}$$

where

$$\left.\begin{aligned} T_{ij}^{(1)} &= \beta_{ij}^{(0)}, \\ T_{ij}^{(2)} &= \beta_{ij}^{(1)} + (\boldsymbol{\beta}^{(0)^2})_{ij}, \\ T_{ij}^{(3)} &= \beta_{ij}^{(2)} + (\boldsymbol{\beta}^{(0)}\cdot\boldsymbol{\beta}^{(1)})_{ij} + 2(\boldsymbol{\beta}^{(1)}\cdot\boldsymbol{\beta}^{(0)})_{ij} + (\boldsymbol{\beta}^{(0)^3})_{ij}, \end{aligned}\right\} \tag{6.2b}$$

and

$$\left.\begin{aligned} \boldsymbol{\beta}^{(0)} &= \boldsymbol{A}^{*\mathrm{T}} + \boldsymbol{\lambda}^{(0)}\cdot(\boldsymbol{\alpha}-\boldsymbol{\alpha}^{\mathrm{T}}), \\ \boldsymbol{\beta}^{(1)} &= \boldsymbol{\lambda}^{(1)}\cdot(\boldsymbol{\alpha}-\boldsymbol{\alpha}^{\mathrm{T}}), \\ \boldsymbol{\beta}^{(2)} &= \boldsymbol{\lambda}^{(2)}\cdot(\boldsymbol{\alpha}-\boldsymbol{\alpha}^{\mathrm{T}}), \end{aligned}\right\} \tag{6.2c}$$

where $\boldsymbol{\lambda}^{(n)}$ are sums of products of nth powers of $\boldsymbol{A}^*$ and polynomials in $\boldsymbol{\kappa}$, given in Appendix E.

The vorticity-moment tensor can then be expressed in terms of the initial vorticity spectrum $\Omega_{pq}(\boldsymbol{\kappa},0)$:

$$\langle\overline{\omega_i\,w_j}\rangle = W_{ij}(t) = \int\langle T_{ip}(\boldsymbol{\kappa},t)\,T_{jq}(\boldsymbol{\kappa},t)\rangle\,\Omega_{pq}(\boldsymbol{\kappa},0)\,\mathrm{d}\boldsymbol{\kappa}. \tag{6.3}$$

If $\boldsymbol{\alpha}$ is isotropic and Gaussian with zero mean, it implies that

$$\langle\boldsymbol{\alpha}\rangle = 0, \qquad \langle\boldsymbol{\alpha\alpha\alpha}\rangle = 0. \tag{6.4}$$

and thence (see Appendix E) that the straining tensor applied force is expressed in terms of the mean-square vorticity Ω_0^2 and the mean square of the deformation of the large-scale turbulence (4.4b). Then $\langle T_{ip}\,T_{jq}\rangle$ is expressed in terms of Ω_0^2, α_0^2 and products of $\boldsymbol{\kappa}$, and thence $W_{ij}(t)$ is derived, for a general mean straining motion, in terms of the principal strains of the mean motion, $A^{(i)}$:

$$\begin{aligned} W_{ij}(t) = {} & W_{ij}(0) + (A^{(i)}+A^{(j)})\,W_{ij}(0)\,t + \tfrac{1}{2}[(A^{(i)}+A^{(j)})^2 W_{ij}(0) \\ & + (\tfrac{11}{5}\alpha_0^2 - 2\Omega_0^2)\,W_{ij}(0) + (\tfrac{3}{5}\alpha_0^2 + \tfrac{2}{3}\Omega_0^2)\,\delta_{ij}\,W_{aa}(0)]\,t^2 \\ & + \tfrac{1}{6}[(A^{(i)}+A^{(j)})^3 W_{ij}(0) + (\tfrac{57}{10}\alpha_0^2 - 5\Omega_0^2)(A^{(i)}+A^{(j)})\,W_{ij}(0) \\ & + (\tfrac{9}{10}\alpha_0^2 + \Omega_0^2)(2A^{(a)}+A^{(i)}+A^{(j)})\,\delta_{ij}\,W_{aa}(0)]\,t^3. \end{aligned} \tag{6.5}$$

As an example we consider a two-dimensional mean straining motion

$$\boldsymbol{A} = \begin{pmatrix} -A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A \end{pmatrix}, \tag{6.6}$$

where $A > 0$, e.g. a two-dimensional stagnation flow. The initial state of the small-scale turbulence is isotropic, so

$$W_{ij}(0) = \omega_0^2 \delta_{ij}. \tag{6.7}$$

It follows from (6.5) that

$$W_{11}(t) = \omega_0^2[e^{-2At} + 2\alpha_0^2 t^2 - (\tfrac{14}{5}\alpha_0^2 - \tfrac{2}{3}\Omega_0^2) At^3 + O(t^4)], \tag{6.8a}$$

$$W_{22}(t) = \omega_0^2[1 + 2\alpha_0^2 t^2 + O(t^4)], \tag{6.8b}$$

$$W_{33}(t) = \omega_0^2[e^{2At} + 2\alpha_0^2 t^2 + (\tfrac{14}{5}\alpha_0^2 - \tfrac{2}{3}\Omega_0^2) At^3 + O(t^4)], \tag{6.8c}$$

and all the other components are identically zero.

The first term reflects the mean vorticity compression/stretching and the second is caused by the random straining. The third is caused by the combined effects of the mean deformation ($\boldsymbol{A}$) and the large-scale random straining ($\boldsymbol{\alpha}$). Note that the effect of random irrotational distortions is to amplify the anisotropy induced by the mean straining. The explanation is that the vortex lines stretched in the x_3 direction by the mean motion experience a net increase in stretching when they experience a random distortion, while the vortex lines in the x_1 direction experience a net compression. To understand this mathematically consider the random motion to be rectilinear with the mean straining $e^{\pm At}$, i.e. to be either $e^{\alpha t}$ or $e^{-\alpha t}$, so that the lengths l_3 and l_1 are

$$l_3 = \tfrac{1}{2}(e^{\alpha t} + e^{-\alpha t})\, e^{At} \sim e^{At} + \tfrac{1}{2}\alpha^2 t^2 + \tfrac{1}{2}A\alpha^2 t^3 + \dots,$$

$$l_1 = \tfrac{1}{2}(e^{\alpha t} + e^{-\alpha t})\, e^{-At} \sim e^{-At} + \tfrac{1}{2}\alpha^2 t^2 - \tfrac{1}{2}A\alpha^2 t^3 + \dots.$$

The term in t^3 gives the interaction effect.

The effect of rotational motion is clearly to reduce the anisotropy by rotating the highly stretched vortex lines l_3 to give a component in the x_1 direction – depicted in figure 2.

It is possible that these ideas are applicable to turbulent shear flows under the influence of external turbulence or sound fields. The main distortion in a shear flow is an irrotational strain, and some of the effects of external turbulence or acoustic forcing are irrotational. Perhaps the well-known synergistic effects in this interaction (which have never been adequately analysed) are partly explored by this calculation.

Our result agrees in general with the result established by Bertoglio (1986) that the transfer of energy between wavenumbers is qualitatively different in straining flows. He found that, as the velocity-spectrum tensor is changed, so is the tensor for the transfer of energy.

In our second analysis of this interaction we consider the sequential application of a mean strain $\boldsymbol{A}$ and then a random strain $\boldsymbol{\alpha}$. The former is the usual r.d.t. calculation and the latter is the same as in §4.1. So we can calculate the changes in the velocity field as well as the vorticity field:

$$\boldsymbol{A} = \begin{pmatrix} -A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A \end{pmatrix}, \quad \boldsymbol{\alpha} = 0 \quad \text{for} \quad 0 \leqslant t \leqslant t_1,$$

and

$$\left.\begin{aligned} \boldsymbol{A} = 0, \quad \langle \boldsymbol{\alpha} \rangle = \langle \boldsymbol{\alpha\alpha\alpha} \rangle = 0, \\ \langle \alpha_{ij}\alpha_{kl} \rangle = -\tfrac{1}{5}\alpha_0^2 \delta_{ij}\delta_{kl} + (\tfrac{3}{10}\alpha_0^2 + \tfrac{1}{3}\Omega_0^2)\delta_{ik}\delta_{jl} + (\tfrac{3}{10}\alpha_0^2 - \tfrac{1}{3}\Omega_0^2)\delta_{il}\delta_{jk}, \end{aligned}\right\} \quad \text{for} \quad t_1 \leqslant t.$$

The changes in the vorticity moments are given by:
for mean distortion $0 \leqslant t \leqslant t_1$

$$W_{11}(t) = \omega_0^2 \mathrm{e}^{-2At}, \tag{6.9a}$$

$$W_{22}(t) = \omega_0^2, \tag{6.9b}$$

$$W_{33}(t) = \omega_0^2 \mathrm{e}^{2At}, \tag{6.9c}$$

$$W_{ll}(t) = 3\omega_0^2(1+\tfrac{4}{3}(At)^2+\ldots); \tag{6.9d}$$

for turbulence interaction $t \geqslant t_1$

$$W_{11}(t) = \omega_0^2\left\{\mathrm{e}^{-2\tau_{A1}}+\left[2-\left(\frac{11}{5}-\frac{2\Omega_0^2}{\alpha_0^2}\right)\tau_{A1}\right]\tau_\alpha^2+\ldots\right\}, \tag{6.10a}$$

$$W_{22}(t) = \omega_0^2[1+2\tau_\alpha^2+\ldots], \tag{6.10b}$$

$$W_{33}(t) = \omega_0^2\left\{\mathrm{e}^{2\tau_{A1}}+\left[2+\left(\frac{11}{5}-\frac{2\Omega_0^2}{\alpha_0^2}\right)\tau_{A1}\right]\tau_\alpha^2+\ldots\right\}, \tag{6.10c}$$

$$W_{ll}(t) = 3\omega_0^2[1+\tfrac{4}{3}\tau_{A1}^2+2\tau_\alpha^2+O(\tau_{A1}^3,\tau_\alpha^3)], \tag{6.10d}$$

where
$$\tau_\alpha = \alpha_0(t-t_1), \qquad \tau_{A1} = At_1. \tag{6.11}$$

The anisotropy tensor of vorticity for $t \geqslant t_1$ is given by

$$a_{11}(t) = -\frac{1}{3}\left[2\tau_{A1}-\tfrac{2}{3}\tau_{A1}^2-\left(\frac{9}{5}+\frac{2\Omega_0^2}{\alpha_0^2}\right)\tau_{A1}\tau_\alpha^2\right], \tag{6.12a}$$

$$a_{22}(t) = -\tfrac{4}{9}\tau_{A1}^2, \tag{6.12b}$$

$$a_{33}(t) = \frac{1}{3}\left[2\tau_{A1}+\tfrac{2}{3}t_{A1}^2-\left(\frac{9}{5}+\frac{2\Omega_0^2}{\alpha_0^2}\right)\tau_{A1}\tau_\alpha^2\right]. \tag{6.12c}$$

The changes in the velocity moments are given in terms of the initial isotropic small-scale turbulence $B_{ij}(0) = \overline{u_i u_j}(0) = u_0^2\,\delta_{ij}$:
for mean distortion $0 \leqslant t \leqslant t_1$

$$B_{11}(t) = u_0^2(1+\tfrac{4}{5}At+\tfrac{12}{35}(At)^2+\ldots), \tag{6.13a}$$

$$B_{22}(t) = u_0^2(1+\tfrac{32}{35}(At)^2+\ldots), \tag{6.13b}$$

$$B_{33}(t) = u_0^2(1-\tfrac{4}{5}At+\tfrac{12}{35}(At)^2+\ldots), \tag{6.13c}$$

$$B_{ll}(t) = 3u_0^2(1+\tfrac{8}{15}(At)^2+\ldots); \tag{6.13d}$$

for turbulence interaction $t \geqslant t_1$

$$B_{11}(t) = u_0^2\left\{1+\tfrac{4}{5}\tau_{A1}+\tfrac{12}{35}\tau_{A1}^2+\frac{4}{5}\left[1-\left(\frac{1}{10}+\frac{\Omega_0^2}{\alpha_0^2}\right)\tau_{A1}\right]\tau_\alpha^2+\ldots\right\}, \tag{6.14a}$$

$$B_{22}(t) = u_0^2\{1+\tfrac{32}{35}\tau_{A1}^2+\tfrac{4}{5}\tau_\alpha^2+\ldots\}, \tag{6.14b}$$

$$B_{33}(t) = u_0^2\left\{1-\tfrac{4}{5}\tau_{A1}+\tfrac{12}{35}\tau_{A1}^2+\frac{4}{5}\left[1+\left(\frac{1}{10}+\frac{\Omega_0^2}{\alpha_0^2}\right)\tau_{A1}\right]\tau_\alpha^2+\ldots\right\}, \tag{6.14c}$$

$$B_{ll}(t) = 3u_0^2\{1+\tfrac{8}{15}\tau_{A1}^2+\tfrac{4}{5}\tau_\alpha^2+\ldots\}. \tag{6.14d}$$

The anisotropy tensor of velocity for $t \geqslant t_1$ is given by

$$b_{11}(t) = \frac{1}{3}\left[\tfrac{4}{5}\tau_{A1} - \tfrac{4}{21}\tau_{A1}^2 - \left(\frac{18}{25} + \frac{4\Omega_0^2}{5\alpha_0^2}\right)\tau_{A1}\tau_\alpha^2\right], \tag{6.15a}$$

$$b_{22}(t) = \tfrac{8}{63}\tau_{A1}^2, \tag{6.15b}$$

$$b_{33}(t) = -\frac{1}{3}\left[\tfrac{4}{5}\tau_{A1} + \tfrac{4}{21}\tau_{A1}^2 - \left(\frac{18}{25} + \frac{4\Omega_0^2}{5\alpha_0^2}\right)\tau_{A1}\tau_\alpha^2\right]. \tag{6.15c}$$

In this case the isotropic large-scale straining is only applied after the turbulence has been distorted. As in the result of §4.1, the anisotropy decreases in proportion to τ_α^2 (here τ_α^2 is measured from the time when the interaction with the large scale begins). Both irrotational and rotational strains act to reduce the anisotropy tensor a_{ij}. But only if $\Omega_0^2 > \frac{11}{10}\alpha_0^2$ is the absolute difference in the mean-square vorticity in the x_1 and x_3 directions reduced (i.e. $W_{33} - W_{11}$). (Since both W_{11} and W_{33} increase on account of the term $2\tau_\alpha^2$, even if $W_{33} - W_{11}$ increases, $|a_{11}|$ and $|a_{33}|$ decrease!) In reality the term $2\tau_\alpha^2$ is compensated by the loss of energy by dissipation, so the interesting term to look for in measurements or computations is $W_{33} - W_{11}$.

On the other hand, the initial difference between the mean square of the velocity components is relatively smaller and is reduced whatever the relative magnitude of Ω_0^2 and α_0^2. It is also reduced faster.

7. Concluding remarks

In the Introduction we raised some general questions about the interactions between large- and small-scale turbulence. Our analysis enables us to give the following answers.

(i) *Energy transfer*

These interactions occur through the straining and rotation of the vorticity of the small-scale by the large-scale turbulence. The energy of the small scale initially increases purely as a result of the irrotational straining motions. The rotational distortion does not affect the energy initially. At later times the distortions by these two processes interact. It is likely then that the rotational distortion will reduce the energy transfer, though this has not been proved.

We note that the initial rate of increase of kinetic energy of the small scale is proportional to the square of time and that the timescale for this increase is α_0^{-1}. (As with turbulent flows in mean strains, it is not clear whether this rate of increase will drop to zero at later times.)

We also note that the initial rate of increase of mean kinetic energy $q^2(t)$ is $\frac{2}{5}$ of the rate of increase of the mean-square vorticity $\overline{\omega^2}$. Under the action of large-scale random irrotational straining, $\overline{q^2(t)}$ increases at an exponential rate that is a quarter of the exponential increase of $\overline{\omega^2}$.

(ii) *Anisotropy*

(*a*) When the small scale is anisotropic and the large scale is isotropic, the interaction leads to the small-scale anisotropy (measured by the tensor b_{ij}) decreasing as

$$\frac{\mathrm{d}b_{ij}}{\mathrm{d}t} \propto -tb_{ij}\,(\text{straining})^2.$$

This rate of reduction is proportional to the local value of anisotropy; it is about equally reduced by irrotational and rotational straining; and initially it is independent of the anisotropy in wavenumber space of the small-scale energy. It is equal to the rate of reduction of the anisotropy of the vorticity moments. But over large times we have found that the irrotational straining cannot lead to an isotropic velocity or vorticity field (though the former is more isotropic than the latter). The distribution of small-scale energy in wavenumber space, defined by a new tensor c_{ij}, becomes equally as important as the velocity-moment anisotropy tensor b_{ij}. In fact, if initially, the velocity tensor is isotropic but the wavenumber distribution is anisotropic (i.e. $b_{ij} = 0$ and $c_{ij} \neq 0$), then the small-scale turbulence becomes anisotropic at later times. We have shown analytically (albeit for a special case) why the rate of reduction of the anisotropy tensor cannot be a unique function of the tensor, in a given type of turbulence.

(*b*) If the large scale is anisotropic, the interaction leads to the small scales becoming anisotropic. The rate of change of the anisotropy of the small scale is

$$\frac{\mathrm{d}b_{ij}}{\mathrm{d}t} \propto -t\ (\text{anisotropic strain rate})^2.$$

Where the anisotropic strain rate is large (e.g. near an interface), this can lead to large changes in the small-scale anisotropy. (The anisotropy of higher-order statistics of the small scales may also be significantly affected, as certainly occurs in turbulent thermal convection – Hunt, Kaimal & Gaynor 1988.)

(iii) *Spectra*

The wavenumber spectrum of the small-scale turbulence is broadened by random straining. This is a relatively fast process; and the peak $E_{\max}$ at $k = k_0$, of the small-scale spectrum initially decreases as

$$E_{\max} \propto \frac{q_0^2}{k_0 \alpha_0 t}.$$

The width of the spectrum increases in proportion to $k_0 \alpha_0 t$, or $U't/L_0 l_0$ if l_0 and L_0 are the scales of the small- and large-scale turbulence.

These results show that the rate of deformation of the spectrum of the small-scale turbulence is greater for the smaller scales. At large times the result (5.9) indicates that the peak of $E(\chi, t)$ moves to a higher or lower wavenumber depending on whether the large scale motion is elongating or squashing.

(iv) *The effects of mean distortion on the interaction between large- and small-scale turbulence*

We have shown that in an irrotational mean straining motion, the large-scale eddies induce additional irrotational straining, in such a way as to amplify the anisotropy of small scales induced by the mean strain. Although the rotational strains reduce this anisotropy, their effect is weaker than that of the irrotational strains if the straining is isotropic. This suggests that the nature of the large-scale turbulence or straining may have a significant effect on the smaller scales in straining flows. The result also indicates a reason why in some distorting turbulent flows, the nonlinear processes are not particularly effective in limiting the distortion by the mean flow.

Following a period of mean strain when the turbulence is distorted, isotropic large-scale eddies induce the smaller-scale eddies to return to isotropy by both irrotational and rotational straining.

(v) *Modelling*

(*a*) We have shown that interaction between large and small scales of turbulence is initially proportional to the time of interaction, e.g.

$$\frac{\mathrm{d}b_{ij}}{\mathrm{d}t} \propto -tb_{ij}\,(\text{straining})^2.$$

As explained in §4.2, in current second-order one-point models and in more complex spectral models containing some relaxation process, this result would be

$$\frac{\mathrm{d}b_{ij}}{\mathrm{d}t} \propto \frac{b_{ij}}{(\text{time scale})} \sim b_{ij}\,(\text{strain rate}).$$

Thus our computation shows that an interaction between scales begins more slowly than predicted by the simple relaxation models. (By analogy, in turbulent diffusion the rate of increase of variance of particle displacements $\mathrm{d}\sigma^2/\mathrm{d}t$ is initially proportional to t, and only after an integral scale is it proportional to T_L; thus eddy diffusion models, in which it is assumed that $\mathrm{d}\sigma^2/\mathrm{d}t$ is proportional to T_L for all t, can only be useful when $t \gtrsim T_L$. In the same way relaxation models are only appropriate for changes over timescales greater than T_L, (J. Herring, private communication.)

(*b*) In most models of one-point second-order moments, the pressure–strain correlations

$$(P_{\mathrm{PG}})_{ij} = \frac{1}{\rho}\overline{p\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)}$$

are assumed to be proportional to the anisotropy tensor of the velocity moments b_{ij}. We have shown that initially they should be proportional to time and to the difference between the usual anisotropy velocity-moment tensor b_{ij} and the wavenumber anisotropy tensor c_{ij}. The latter point is consistent with the heuristic calculations of this expression by Weinstock & Burk (1985) who find that (P_{PG}/b_{ij}) is sensitive to the anisotropy of turbulence. They appear not to have considered the possibility of the sensitivity to the wavenumber anisotropy which is related to c_{ij}.

In the models for the production of turbulence by Reynolds stress,

$$(P_{\mathrm{RS}})_{ij} = -\left(\overline{u_i u_k}\frac{\partial U_j}{\partial x_k}+\overline{u_j u_k}\frac{\partial U_i}{\partial x_k}\right),$$

these effects of time dependence and wavenumber anisotropy are also omitted. This means that in some calculations these models must overpredict the return to isotropy because of the neglect of the 't' dependence. But the neglect of c_{ij} does not matter initially because the c_{ij} terms in P_{PG} and P_{RS} cancel out. At larger times, however, c_{ij} probably does play a part and that needs to be considered.

(vi) *Further reservations*

There are several important mechanisms for transfer between wavenumbers that have not been considered in these calculations.

(*a*) The interaction between wavenumbers within the range of wavenumbers in the small-scale turbulence has a large effect on the tendency towards isotropy of this component of turbulence, if $u'/l > U'/L$, as in a typical continuous spectrum, at very high Reynolds number. The analysis here indicates the additional effects by large-scale straining.

(*b*) The transfer of energy between wavenumbers can also occur by a pure advection process. In two-dimensional turbulence there is no vortex stretching, but there is a distortion of the wavenumber of the small-scale turbulence by the large-scale turbulence. This process is modelled in the theory presented here. In two-dimensional turbulence high wavenumbers are compressed but no extra vorticity is generated, and so there is no extra energy. Consequently the spectrum of the small-scale turbulence is distorted towards low wavenumbers.

(*c*) We have not considered dissipative processes and the cascade of energy between wavenumbers in the small-scale turbulence. This will be affected by large-scale straining.

Finally we hope that these model calculations will stimulate some further direct computations and further experiments on the interactions between large-scale and small-scale turbulent flows with a distinct separation of scales.

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Copies of Appendices A–E referred to in the text may be obtained on request to the authors or the Editor.

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